**Lecture 13**

**Learning Objectives**

At the end of this class, students should be able to:

- understand the concept of limit
- evaluate the limit

**13.1 Limits**

The value of the function $f(x) = \frac{5x}{2x+3}$ at $x = 1$ is $f(1) = \frac{5\times1}{2\times1+3} = 1$.

Here we can easily find the value of the function just by substituting $x = 1$. Sometimes we can't work something out directly but we can see what it should be as we get closer and closer to that particular value from left hand side and right-hand side.

Let us try to understand the situation with the help of the following illustration.

*Illustration*

Determine the value of the function $f(x) = \frac{x^2-1}{x-1}$ at $x = 1$.

*Solution*

Here $f(1) = \frac{1^2-1}{1-1}$

$= \frac{1-1}{1-1}$

$= \frac{0}{0}$

We don't know the value of 0/0 (it is "indeterminate"), so we need another way of answering this. So instead of trying to work it out for $x = 1$, let's try approaching it closer and closer to 1 from both sides.

Let us prepare tables of values of $x$ approaching 1 and corresponding values of $f(x) = \frac{x^2-1}{x-1}$.

Approaching $x = 1$ from left

| $x$ | 0.5 | 0.9 | 0.99 | 0.999 | 0.99990 | 0.99999 |
|---|---|---|---|---|---|---|
| $f(x) = \frac{x^2-1}{x-1}$ | 1.5 | 1.9 | 1.99 | 1.999 | 1.99990 | 1.99999 |

Approaching $x = 1$ from right

| $x$ | 1.5 | 1.1 | 1.01 | 1.001 | 1.0001 | 1.00001 |
|---|---|---|---|---|---|---|
| $f(x) = \frac{x^2 - 1}{x - 1}$ | 2.5 | 2.1 | 2.01 | 2.001 | 2.0001 | 2.00001 |

Now we see that as $x$ gets close to 1, then $f(x) = \frac{x^2-1}{x-1}$ gets close to 2.

We are now faced with the situation;

When $x = 1$ we don't know the answer (it is indeterminate). But we can see that it is going to be 2.

We want to give the answer "2" but can't, so instead mathematicians say exactly what is going on by using the special word "limit".

The limit of $f(x) = \frac{x^2 - 1}{x - 1}$ as $x$ approaches 1 is 2.

It is written in symbols as: $\lim_{x \to 2} \frac{x^2 - 1}{x - 1} = 2$

So, it is a special way of saying, "ignoring what happens when we get there, but as we get closer and closer the answer gets closer and closer to 2".

If a function $f(x)$ approaches a limit L as $x$ approaches $c$, we say that the limit of a function is L. Mathematically, it is written as

$$\lim_{x \to c} f(x) = L$$

Thus, for the limit of a function to exist as the independent variable approaches $c$, the left-hand and right-hand limits must be equal.

To state it more precisely:

$$\lim_{x \to c} f(x) = L$$

If and only if

$$\lim_{x \to c^-} f(x) = L \quad \text{and} \lim_{x \to c^+} f(x) = L.$$

*Illustration*

Let $f(x) = \begin{cases} x - 1 & \text{if } x \le 3 \\ -2x + 8 & \text{if } x > 3 \end{cases}$. Evaluate $\lim_{x \to 3} f(x)$, if it exists.

*Solution*

Here, left hand limit is

$$\lim_{x \to 3^-} f(x) \quad = \lim_{x \to 3} (x - 1) \qquad [\because f(x) = x - 1 \text{ when } x < 3]$$

$$= 3 - 1 \; = 2$$

Similarly, right hand limit is

$\lim_{x\to 3^+} f(x) \quad = \lim_{x\to 3}(-2x+8) \qquad [\because f(x) = -2x+8 \text{ when } x > 3]$

$= -2 \times 3 + 8 \quad = 2$

Since $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$, therefore, $\lim_{x\to 3} f(x)$ exists.

The following properties of limits, which we list without proof, enable us to evaluate limits of functions algebraically.

## 13.2 Properties of Limits

Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ then

1. $\lim_{x\to a}[f(x)]^r = \left[\lim_{x\to a} f(x)\right]^r = L^r$ — $r$ and $a$ are real numbers
2. $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x) = cL$ — $c$ and $a$ are real numbers
3. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$
4. $\lim_{x\to a}[f(x)g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right] = LM$
5. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}$ — Provided $M \neq 0$
6. For every c in the in the trigonometric function's domain,

   $\lim_{x\to c} \sin x = \sin c$ $\qquad$ $\lim_{x\to c} \operatorname{cosec} x = \operatorname{cosec} c$

   $\lim_{x\to c} \cos x = \cos c$ $\qquad$ $\lim_{x\to c} \sec x = \sec c$

   $\lim_{x\to c} \tan x = \tan c$ $\qquad$ $\lim_{x\to c} \cot x = \cot c$
7. $\lim_{x\to 0} \frac{\sin x}{x} = 1$

*Illustration*

Evaluate $\lim_{x\to 3} \frac{2x^2+10}{x+2}$

*Solution*

$$\lim_{x\to 3} \frac{2x^2+10}{x+2} = \frac{\lim_{x\to 3}(2x^2+10)}{\lim_{x\to 3}(x+2)}$$

$$= \frac{2(3)^2+10}{3+2}$$

$$= \frac{28}{5}$$

### 13.3 Limits of Polynomial and Rational Functions

Let $p(x)$ be a polynomial function, $c$ is any real number. Then

$$\lim_{x\to c} p(x) = p(c)$$

*Illustration*

Evaluate $\lim_{x\to 3}(5x^3 - 6x^2 + 4x - 1)$

*Solution*

$$\lim_{x\to 3}(5x^3 - 6x^2 + 4x - 1) = 5(3)^3 - 6(3)^2 + 4(3) - 1$$

$$= 135 - 54 + 12 - 1$$

$$= 92$$

Let $r(x) = p(x)/q(x)$ be a rational function, where $p(x)$ and $q(x)$ are polynomials. Let $c$ be a number such that $q(c) \neq 0$. Then

$$\lim_{x\to c} r(x) = r(c)$$

*Illustration*

Evaluate $\lim_{x\to 2} \frac{x+2}{x-2}$

*Solution*

$$\lim_{x\to 2} \frac{x+2}{x-2} = \frac{2+2}{2-2}$$

$$= \frac{4}{0}$$

$$= \infty$$

Thus, the limit does not exist.

*Illustration*

Evaluate $\lim_{x\to 2} \frac{3(x^2-4)}{x-2}$.

*Solution*

The function $\frac{3(x^2-4)}{x-2}$ is not defined when $x = 2$, since $\frac{3(x^2-4)}{x-2} = \frac{0}{0}$ which is an indeterminate form.

The expression $\frac{0}{0}$ does not provide us with a solution to our problem. The following technique can be used to solve this type of problems.

1. Replace the given function with an appropriate one that takes on the same values as the original function everywhere except at $x = a$.

2. Evaluate the limit of this function as $x$ approaches $a$.

We can write

$$\frac{3(x^2-4)}{x-2} = \frac{3(x-2)(x+2)}{(x-2)}$$

which, upon canceling the common factors, is equivalent to $3(x+2)$, provided $x \neq 2$.

Thus, we replace $\frac{3(x^2-4)}{x-2}$ with $3(x+2)$ and find that

$$\lim_{x \to 2} \frac{3(x^2-4)}{x-2} = \lim_{x \to 2} 3(x+2)$$

$$= 3(2 + 2)$$

$$= 12$$

*Illustration*

Evaluate $\lim\limits_{x \to 0} \frac{3x \tan x}{\sin x}$

*Solution*

$$\lim_{x \to 0} \frac{3x \tan x}{\sin x} = \lim_{x \to 0} \frac{3x \sin x}{\sin x \times \cos x}$$

$$= \lim_{x \to 0} \frac{3x}{\cos x}$$

$$= \frac{3 \times 0}{\cos 0} = \frac{0}{1} = 0$$

*Illustration*

Evaluate $\lim\limits_{x \to 0} \frac{\sin^2 x}{x}$

*Solution*

$$\lim_{x \to 0} \frac{\sin^2 x}{x} = \lim_{x \to 0} \frac{\sin x \times \sin x}{x}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \times \lim_{x \to 0} \frac{\sin x}{1}$$

$$= 1 \times 0 = 0$$

**13.4 Approaching Infinity**

Let us try to understand the concept with the help of the following calculation.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| x | 1 | 2 | 5 | 10 | 100 | 1000 | 10000 | 100000 | 1000000 |
| 1/x | 1 | 0.5 | 0.2 | 0.1 | 0.01 | 0.001 | 0.0001 | 0.00001 | 0.000001 |

Here, we see that as x gets larger, 1/x tends towards 0.

We can't say what happens when x gets to infinity but we can see that as x gets larger and larger 1/x is going towards 0.

Thus, the limit of 1/x as x approaches infinity is 0. Mathematically,

$$\lim_{x \to \infty}\left(\frac{1}{x}\right) = 0$$

It is a mathematical way of saying "we are not talking about when x = ∞, but we know as x gets bigger, the answer gets closer and closer to 0".

*Illustration*

Evaluate $\lim_{x \to \infty}\left(\frac{1}{x-1}\right)$

*Solution*

Here $\lim_{x \to \infty}\left(\frac{1}{x-1}\right)$

$= \frac{1}{\infty - 1}$

$= \frac{1}{\infty}$

$= 0$

**Exercise for Reader**

1. Evaluate the following limits

a) $\lim_{x \to 3} 2$

b) $\lim_{x \to -1} (-5)$

c) $\lim_{x \to 2} x$

d) $\lim_{x \to -3} (-7x)$

2. Let $f(x) = \begin{cases} 1 - x^2 & \text{if } x \leq 0 \\ 1 + x^2 & \text{if } x > 0 \end{cases}$. Find

   a) $\lim_{x \to 0^-} f(x)$ b) $\lim_{x \to 0^+} f(x)$ c) $\lim_{x \to 0} f(x)$ d) f (0)

3. Let $f(x) = \begin{cases} -2x + 4 & \text{if } x < 1 \\ 4 & \text{if } x = 1 \\ x^2 + 1 & \text{if } x > 1 \end{cases}$. Evaluate $\lim_{x \to 1} f(x)$, if it exists.

4. Evaluate the following limits.

   a) $\lim_{x \to 0} \frac{\tan(5x)}{\text{Sin}(2x)}$

   b) $\lim_{x \to 0} \frac{\sin(4x).\tan(5x)}{3x^2}$

   c) $\lim_{x \to \infty} 5x^2 + 6$