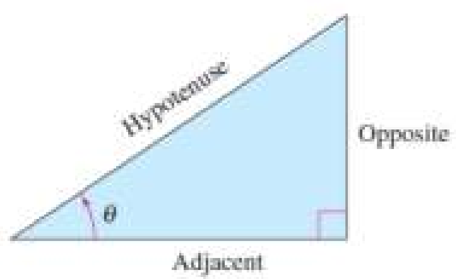
Lecture 12

Learning Objectives:
At the end of this class, students should be able to:
- familiar with trigonometric functions
- plot the graph of trigonometric functions
- solve the problems related to trigonometric functions

Trigonometric functions
Trigonometric ratios are defined for acute angles as the ratio of the sides of a right-angled triangle. A triangle is a right triangle if one of its angles is a right angle. The trigonometric functions are defined as ratios of two sides of a right triangle.

If θ is any acute angle of a right triangle, as shown in the following figure.



For every angle θ in the triangle, there is the side of the triangle adjacent to it (from here on denoted as "Adjacent"), the side opposite of it (from here on denoted as "Opposite"), and the hypotenuse (from here on denoted as "Hypotenuse"), which is the longest side of the triangle located opposite of the right angle.

For angle θ, the trigonometric functions sine, cosine and tangent of θ are defined as:

$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$$
$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\text{Opposite}}{\text{Adjacent}}$$

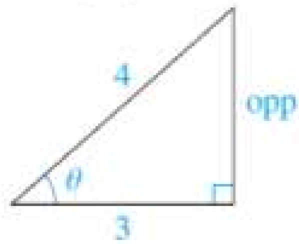
The reciprocal trigonometric functions secant, cosecant and cotangent are defined as:

$$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{Hypotenuse}}{\text{Adjacent}}$$
$$\text{cosec}\theta = \frac{1}{\sin\theta} = \frac{\text{Hypotenuse}}{\text{Opposite}}$$
$$\cot\theta = \frac{1}{\tan\theta} = \frac{\text{Adjacent}}{\text{Opposite}}$$

Illustration
If θ is an acute angle and $\cos\theta = \frac{3}{4}$, find the values of the trigonometric functions of θ.

Solution

We have $\cos\theta = \frac{3}{4}$. Let us plot a right triangle having an acute angle θ with adjacent = 3 and hypotenuse = 4, as shown in the following figure.



According to Pythagorean theorem,

$$(3)^2 + (\text{Opposite})^2 = (4)^2$$

or, $(\text{Opposite})^2 = 16 - 9 = 7$

or, $\text{Opposite} = \sqrt{7}$

Applying the definition of the trigonometric functions of an acute angle of a right triangle, we obtain the following:

$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{\sqrt{7}}{4}$$

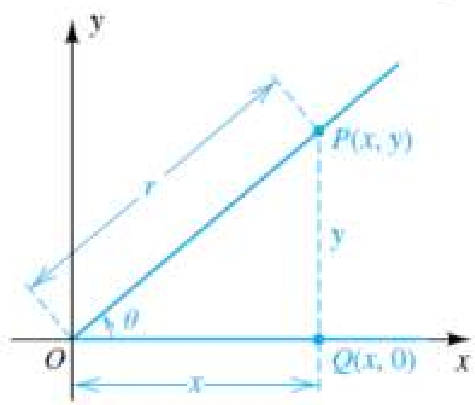
$$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{\sqrt{7}}{3}$$

$$\sec\theta = \frac{\text{Hypotenuse}}{\text{Adjacent}} = \frac{4}{3}$$

$$\text{cosec}\theta = \frac{\text{Hypotenuse}}{\text{Opposite}} = \frac{4}{\sqrt{7}}$$

$$\cot\theta = \frac{\text{Adjacent}}{\text{Opposite}} = \frac{3}{\sqrt{7}}$$

Let P(x, y) be any point on the a rectangular coordinate system such that the line OP makes an acute angle θ with the positive x-axis as shown in the following figure.



Since O is the origin, so OQ = x, PQ = y, and OP = $r = \sqrt{x^2 + y^2}$. Thus, for right triangle OQP, we have

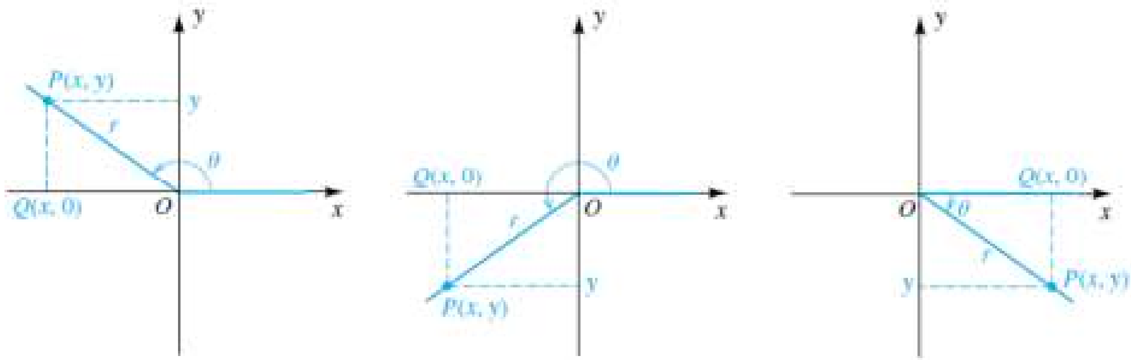
$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{y}{r}$$

$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{x}{r}$$

$$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{y}{x}$$

Since many applied problems involve angles that are not acute, it is necessary to extend the definition of the trigonometric functions. We now consider angles of the types illustrated in the following figures (or any other angle, either positive, negative, or zero). Note that in the following figures, the value of *x* or *y* may be negative. In each case, side QP has length $|y|$, side OQ has length $|x|$, and the hypotenuse OP has length *r*.

We shall define the six trigonometric functions so that their values agree with those given previously whenever the angle is acute. It is understood that if a zero denominator occurs, then the corresponding function value is undefined.



Let θ be an angle in standard position on a rectangular coordinate system. Let P(*x*, *y*) be any point other than the origin O on the a rectangular coordinate system such that the line OP makes an angle θ with positive *x*-axis as shown in the above figures.
If OP = $r = \sqrt{x^2 + y^2}$, then

$$\sin\theta = \frac{y}{r}, \cos\theta = \frac{x}{r}, \tan\theta = \frac{y}{x} \quad (\text{if } x \neq 0)$$
$$\text{cosec}\theta = \frac{r}{y} \quad (\text{if } y \neq 0), \sec\theta = \frac{r}{x} \quad (\text{if } x \neq 0), \cot\theta = \frac{x}{y} \quad (\text{if } y \neq 0)$$

The domains of the sine and cosine functions consist of all angles θ. However, $\tan\theta$ and $\sec\theta$ are undefined if $x = 0$ (that is, if the terminal side of θ is on the *y*-axis). Thus, the domains of the tangent and the secant functions consist of all angles except those of radian measure $\left(\frac{\pi}{2}\right) + \pi n$ for any integer *n*.

Some special cases are $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$, and $\pm\frac{5\pi}{2}$. The corresponding degree measures are $\pm 90^o$, $\pm 270^o$, and $\pm 450^o$.

The domains of the cotangent and cosecant functions consist of all angles except those that have *y* = 0 (that is, all angles except those having terminal sides on the *x*-axis). These are the angles of radian measure πn (or degree measure $180^o . n$) for any integer *n*.

The following table summarize the domain of each function, where *n* denotes any integer.

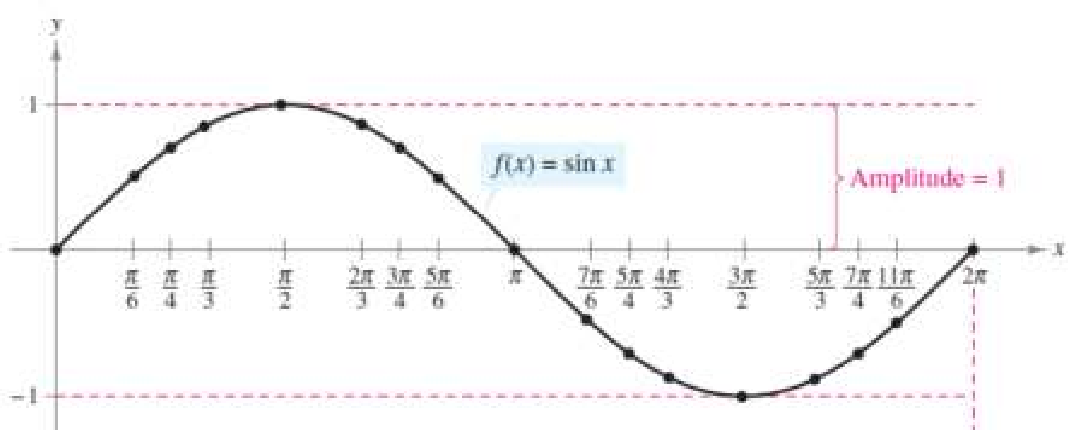
Function		Domain
sine,	cosine	every angle θ
tangent,	secant	every angle θ except $\theta = \frac{\pi}{2} + \pi n = 90^\circ + 180^\circ \cdot n$
cotangent,	cosecant	every angle θ except $\theta = \pi n = 180^\circ \cdot n$

For any point P(x, y) in the above cases, $|x| \leq r$ and $|y| \leq r$, equivalently, $|x/r| \leq 1$ and $|y/r| \leq 1$.
Thus
$|\sin\theta| \leq 1, |\cos\theta| \leq 1, |\mathrm{cosec}\theta| \geq 1,$ and $|\sec\theta| \geq 1$ for every θ in the domain of these functions.

Illustration
Plot the graph of the function $f(x) = \sin x$.
Solution
Let us list the coordinates (x, $f(x)$) of some points on the graph of $f(x) = \sin x$ in tabular form, as follows.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin x$	0.00	0.50	0.71	0.87	1.00	0.87	0.71	0.50	0.00

The amplitude of the sine function (or the cosine function) is defined to be half of the difference between its maximum and minimum values. So, the amplitude of $f(x) = \sin x$ is 1. The graph of the function $f(x) = \sin x$ as follows.



Exercise for Reader

1. If θ is an acute angle and $\sin\theta = \frac{3}{5}$, find the values of the trigonometric functions of θ.
2. A forester, 200 feet from the base of a redwood tree, observes that the angle between the ground and the top of the tree is 60^o. Estimate the height of the tree.
3. Plot the graph of the function $f(x) = \cos x$.