Lecture 11

Learning Objectives:
At the end of this class, students should be able to:

- solve the exponential equations by using logarithm
- plot the graph of logarithmic functions
- solve the problems related to logarithmic functions

11.1 Logarithms

We know that,

$$3^4 = 81$$

This expression can be written as

$$\log_3 81 = 4$$

In this case the logarithm, 4, is the exponent to which we have to raise the base 3 to obtain 81. In general, if $v = \log_b u$, then *v* is the exponent to which the base *b* must be raised to obtain *u*. Where, $b > 0$ and $b \neq 1$.

The *b* is called the *base* in both $\log_b u = v$ and $b^v = u$, and *v* is the logarithm in $\log_b u = v$ and the exponent in $b^v = u$. Thus, a logarithm is an exponent. Where *b* (≠1) and *u* both are positive real numbers.

The two most commonly used bases for logarithms are base 10 and base e. Logarithms with base 10 are called common logarithms which are defined as $v = \log_{10} u$. The more common way of expressing such logarithms is $v = \log u$. Logarithms with base e are called natural logarithms which are defined as $v = \log_e u$. The more common way of expressing such logarithms is $v = \ln u$.

Logarithmic Properties

Computations involving logarithms are facilitated by the following laws of logarithms.
Let $b > 0$ and $b \neq 1$, and *M* and *N* are positive real numbers, then

- $\log_b b = 1$
- $\log_b 1 = 0$
- $\log_b (b^x) = x$
- $\log_b M = \log_b N$ if and only if $M = N$
- $\log_b (MN) = \log_b M + \log_b N$
- $\log_b (M/N) = \log_b M - \log_b N$
- $\log_b (M^N) = N \log_b M$
- $\log_b x = \dfrac{\log_c x}{\log_c b}$; where $b > 0,\ c > 0$, $b \neq 1$, and $c \neq 1$

11.2 Exponential and Logarithmic Equations

A logarithmic equation is an equation that involves the logarithm of an expression containing an unknown. For example, $3\ln(x-1) = 7$ is a logarithmic equation. On the other hand, an equation having the unknown appearing in an exponent is called the exponential equation. For example, $5^{3x} = 1$ is an exponential equation.
The following illustrations present the solution procedures of Exponential and Logarithmic Equations.

Illustration
Solve the equation: $\log_5(2x+3)=2$.
Solution
We have $\log_5(2x+3)=2$, then

$2x+3=5^2$

or $2x=25-3$

$\therefore$ $x=11$

Illustration
Solve the equation: $150=\dfrac{200}{1+30e^{-0.3x}}$.
Solution
We have $150=\dfrac{200}{1+30e^{-0.3x}}$

or $1+30e^{-0.3x}=\dfrac{200}{150}$

or $30e^{-0.3x}=\dfrac{4}{3}-1$

or $e^{-0.3x}=\dfrac{1}{3}\times\dfrac{1}{30}$

or $e^{0.3x}=90$

Taking base *e* logarithm on both sides, we get

$0.3x\ln e=\ln 90$

or $0.3x=\ln 90$

or $x=\dfrac{\ln 90}{0.3}$

$\therefore$ $x=14.999$ [using calculator]

Illustration
A community grows in such a way that *t* years from now, its population is $P(t)$ thousand, where $P(t) = 51 + 100\ln(t + 3)$

a) What is the population when $t = 0$?
b) How long does it take for the population to double its initial value?

Solution

a) Here, $P(0) = 51 + 100\ln(0 + 3)$
$= 51 + 100\ln 3$
$= 160.861$

b) According to question $P(t) = 2P(0) = 2 \times 160.861$
Using $P(t) = 51 + 100\ln(t + 3)$, we get

$2 \times 160.861 = 51 + 100\ln(t + 3)$

or $270.722 = 100\ln(t + 3)$

or $\ln(t + 3) = 2.70722$

or $t + 3 = e^{2.70722}$

or $t + 3 = 14.98$

or $t = 11.98$

It takes 11.98 years for the population to double its initial value.

11.3 Logarithmic Functions

The inverse of an exponential function is called a logarithmic function. If x is a positive number, then the logarithm of x to the base b (where, $b>0$ and $b\neq1$.), denoted $\log_b x$, is the number y such that by $b^y = x$; that is,

$y = \log_b x$ if and only if by $b^y = x$ for $x > 0$

Illustration

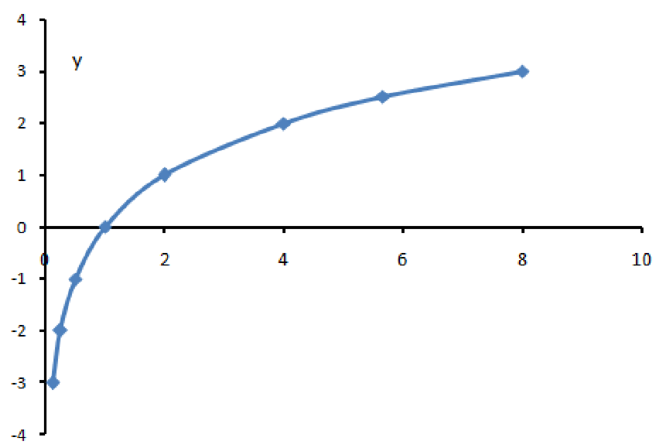
Plot the graph of the function $y = f(x) = \log_2 x$.

Solution

We know that $y = \log_2 x$ is equivalent to $x = 2^y$, thus, finding the required points with the help of $x = 2^y$.

$x = 2^y$	1/8	1/4	1/2	1	2	4	5.65	8
y	-3	-2	-1	0	1	2	2.5	3

The graph of the function $y = \log_2 x$ is as follows:



Basic Properties of the Logarithmic Function

The logarithmic function $y = \log_b x$ ($b>0$ and $b\neq1$) has the following properties.

1. The domain of the function is (0, ∞).
2. The range of the function is (-∞, ∞).
3. The graph passes through the point (1, 0).
4. The graph is a continuous curve on (0, ∞).
5. The graph of the function $f(x) = \log_b x$ increases as x increases.

Illustration

The Richter scale is used to measure the intensity of an earthquake. The magnitude on the Richter scale of an earthquake of intensity I is given by $R = \log\frac{I}{I_o}$ where I is the intensity of the earthquake being measured and I_0 is a certain minimum intensity used for comparison.

a) Find R if I is 15,800,000 times as great as I_0.
b) As of 2010, the 1964 Alaskan earthquake was the most violent U.S. earthquake and the second strongest ever; it measured 9.2 on the Richter scale. Find the intensity of the 1964 Alaskan earthquake.

Solution

a) If $I = 15{,}800{,}000 I_0$ than $I/I_0 = 15{,}800{,}000$. Hence

$R = \log(15{,}800{,}000)$
$= 7.2$ (approximated to 1 decimal place)

b) Substituting $R = 9.2$ in the given expression, we get

$$9.2 = \log \frac{I}{I_o}$$

Rewriting this in exponential form, we get

$$10^{9.2} = \frac{I}{I_0}$$

Using calculator, we get

$$\frac{I}{I_0} = 1{,}580{,}000{,}000 \qquad \text{[approximately]}$$

or, $I = 1{,}580{,}000{,}000 I_0$

Thus, the intensity is 1,580,000,000 times I_0.

Exercise for Reader

1. Solve the following equations
   a) $\log_{25} x = 1/2$
   b) $\log_7 (3x+1) = 2$
   c) $\log_{10} x + \log_{10}(x+1) = \log_{10} 6$
   d) $5(4^{2x-3}) = 120$
   e) $\ln(x+2) + \ln x = \ln(x+12)$
2. On the Richter scale, the magnitude R of an earthquake is given by the formula $R = \log \frac{I}{I_o}$ where I is the intensity of the earthquake being measured and I_0 is the standard reference intensity.
   a) Express the intensity I of an earthquake of magnitude $R = 5$ in terms of the standard intensity I_0.
   b) Express the intensity I of an earthquake of magnitude $R = 8$ in terms of the standard intensity I_0. How many times greater is the intensity of an earthquake of magnitude 8 than one of magnitude 5?
   c) In modern times, the greatest loss of life attributable to an earthquake occurred in Nepal in 2015. Known as the Gorakha earthquake, it registered 7.8 on the Richter scale. How does the intensity of this earthquake compare with the intensity of an earthquake of magnitude $R = 5$?
3. Chemists use the pH (hydrogen potential) of a solution to measure its acidity or basicity. The pH is given by the formula $\text{pH} = -log[\text{H}^+]$, where $[\text{H}^+]$ is the concentration of hydrogen ions in moles per liter.
   a) Most common solutions have pH ranges between 1 and 14. What values of $[\text{H}^+]$ are associated with these extremes?
   b) Find the approximate pH of each of the following:
      i) blood: $[\text{H}^+] = 3.98 \times 10^{-8}$
      ii) beer: $[\text{H}^+] = 6.31 \times 10^{-5}$
      iii) vinegar: $[\text{H}^+] = 6.3 \times 10^{-3}$