

Lecture 10

Learning Objectives:

At the end of this class, students should be able to:

- solve the problems related to piecewise linear functions
- plot the graph of exponential functions
- solve the problems related to exponential functions.

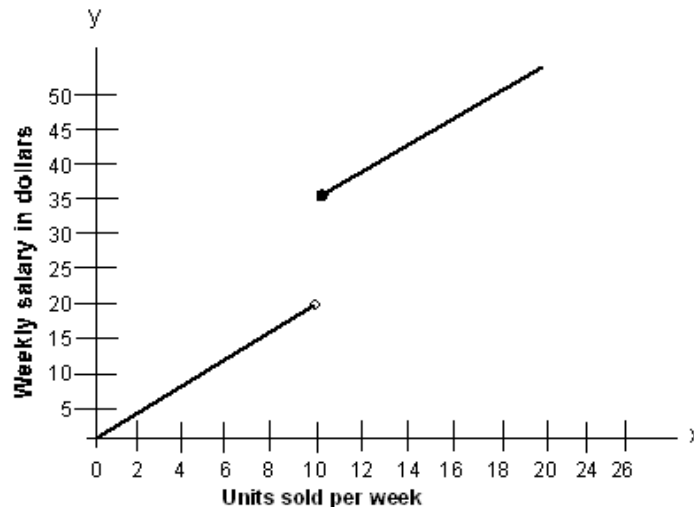
10.1 Piecewise Linear Function

The functional relationship existing between variables is sometimes described by more than one equation. To illustrate, assume that the weekly salary of a salesperson depends upon the number of units sold during the week. When the number of units sold during a week is less than 10, the salesperson receives a commission of \$2 per unit sold. If the number of units sold during a week is 10 or more, a bonus of \$15 will be given. Now the problem is to find the appropriate function representing salesperson's weekly salary.

Assume that y equals a salesperson's weekly salary in dollars and x equals the number of units of a product sold during the week. Then, the required function is given by

$$y = f(x) = \begin{cases} 2x & \text{when } 0 \leq x < 10 \\ 2x + 15 & \text{when } x \geq 10 \end{cases}$$

The graph of the above function is as follows:



Illustration

The monthly charge in dollars for x kilowatt hours (kWh) of electricity used by a residential customer of certain city during the months of November through June is given by the function

$$C(x) = \begin{cases} 10 + 0.094x & \text{if } 0 \leq x \leq 100 \\ 19.40 + 0.075(x - 100) & \text{if } 100 \leq x \leq 500 \\ 49.40 + 0.05(x - 500) & \text{if } x > 500 \end{cases}$$

- What is the monthly charge if 1100 kWh of electricity are consumed in a month?
- What is the monthly charge if 450 kWh are consumed in a month?

Solution

- i) According to the question, $C(1100) = 49.40 + 0.05(1100 - 500) = 79.40$. Thus, the monthly charge = \$79.40.
- ii) When $x = 450$ kWh, $C(450) = 19.40 + 0.075(450 - 100) = \45.65 .

10.2 Exponential Functions

The function defined by $f(x) = b^x$ where $b > 0$ and $b \neq 1$, is called an exponential function with base b and exponent x . The domain of this function is the set of all real numbers. The range of this function is the set of all positive real numbers. For example, $f(x) = 2^x$, $g(x) = 10^x$, and $h(x) = (0.4)^x$ are exponential functions with base 2, 10, and 0.4 respectively.

Illustration

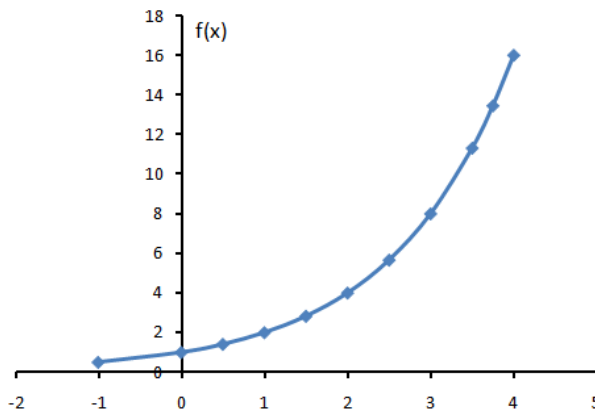
Plot the graph of the function $f(x) = 2^x$

Solution

Let us list the coordinates $(x, f(x))$ of some points on the graph of $f(x) = 2^x$ in tabular form, as follows.

x	-1	0	0.5	1	1.5	2	2.5	3	3.5	3.75	4
$f(x)$	0.50	1.00	1.41	2.00	2.83	4.00	5.66	8.00	11.31	13.45	16.00

The graph of the function $f(x) = 2^x$ is as follows.



From the above figure, we observe that

- the function is defined for all values of x , that is, the domain is the set of real numbers.
- the graph of the function lies entirely above x -axis, that is, the range of the function is the set of positive real numbers.
- $f(x)$ is an increasing function of x .

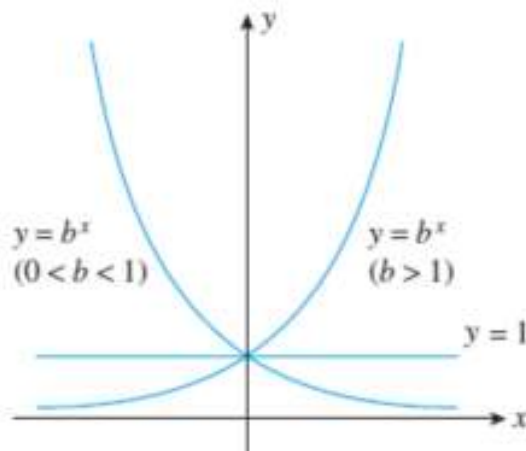
The general shape for the exponential function $f(x) = b^x$ where $b > 1$, depends on the value of b . This class of functions is particularly useful in modeling growth processes.

Illustration

Plot the graph of $y = b^x$ with $b > 1$; $y = b^x$ with $0 < b < 1$; and $y = b^x$ with $b = 1$, i.e., $y = 1$ in the same rectangular coordinate system.

Solution

The required graph is shown in the following figure.



Basic Properties of the Exponential Function

The exponential function $f(x) = b^x$ ($b > 0$ and $b \neq 1$) has the following properties.

1. The domain of the function is $(-\infty, \infty)$.
2. The range of the function is $(0, \infty)$.
3. The graph passes through the point $(0, 1)$.
4. If $b > 1$, then $f(x) = b^x$ increases as x increases.
5. If $0 < b < 1$, then $f(x) = b^x$ decreases as x increases.

Model for Proportional Change

The exponential function is used to model a relationship in which a constant change in the independent variable gives the same proportional change (i.e., percentage increase or decrease) in the dependent variable.

If the change is positive, this is called exponential growth and if it is negative, it is called exponential decay.

For example, because a radioactive substance decays at a rate proportional to the amount of the substance present, the amount of the substance present at a given time can be modeled with an exponential function such as $V = V_0 e^{-kt}$ where V_0 is the initial value (at time $t = 0$) and k is a constant called the decay constant. Also, because the growth rate of a population of bacteria in a petri dish is proportional to its size, the number of bacteria in the dish at a given time can be modeled by an exponential function such as $P = P_0 e^{kt}$ where P_0 is the number of bacteria present initially (at time $t = 0$) and k is a constant called the growth constant.

Illustration

Employers are increasingly turning to GPS (global positioning system) technology to keep track of their fleet vehicles. The estimated number of automatic vehicle trackers installed on fleet vehicles in the United States is approximated by $N(t) = 0.6e^{0.17t}$ ($0 \leq t \leq 5$) where $N(t)$ is measured in millions and t is measured in years, with corresponding to 2010.

- a) What was the number of automatic vehicle trackers installed in the year 2010? How many were projected to be installed in 2015?
- b) Sketch the graph of N .

Solution

a) We have $N(t) = 0.6e^{0.17t}$. The number of automatic vehicle trackers installed in the year 2010 is $N(0) = 0.6e^{0.17 \times 0} = 0.6$ millions, i.e., 600,000.

The number of automatic vehicle trackers installed in the year 2015 is calculated by $N(5) = 0.6e^{0.17 \times 5} = 1.403778811$ millions, i.e., 1,403,778.

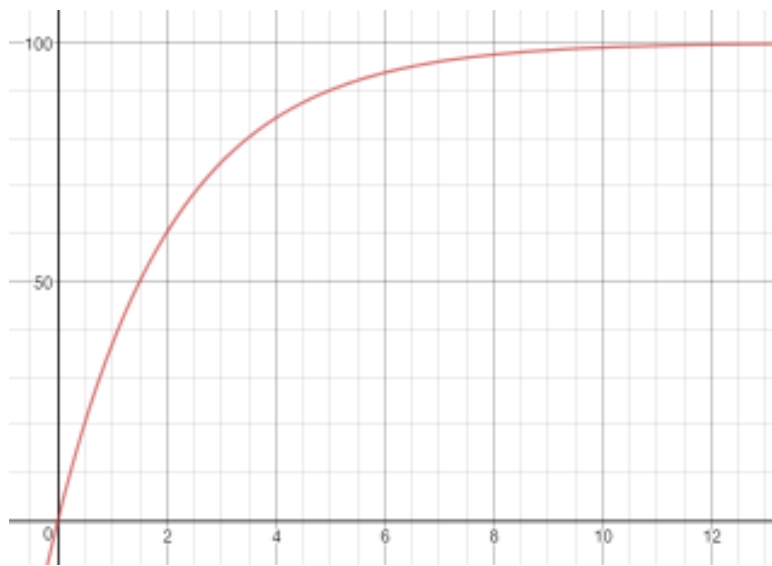
b) This is left for reader.

Illustration

The percent concentration y of a certain drug in the bloodstream at any time t in minutes is given by the function $y = 100(1 - e^{-0.462t})$. Graph this equation for $0 \leq t \leq 10$. Write a sentence that interprets the graph.

Solution

The graph of the given function $y = 100(1 - e^{-0.462t})$ is shown in the following figure



At time 0, the concentration is 0. The concentration rises rapidly for the first 4 minutes and then tends toward 100% as time nears 10 minutes.

Exercise for Reader

1. A single bacterium splits into two bacteria every half hour, so the number of bacteria in a culture quadruples every hour. Thus, the function by which a colony of 10 bacteria multiplies in t hours is given by $y = 10(4^t)$. Graph this equation for $0 \leq t \leq 8$.
2. According to a study conducted in 2000, the projected number of Web addresses (in billions) is approximated by the function $N(t) = 0.45e^{0.5696t}$ ($0 \leq t \leq 5$) where, t is measured in years, with $t = 0$ corresponding to the beginning of 1997.
 - a) Complete the following table by finding the number of Web addresses in each year:

Year	0	1	2	3	4	5
Number of Web Addresses (billions)						

- b) Sketch the graph of N .
3. The number of Internet users in China is projected to be $N(t) = 94.5e^{0.2t}$ ($1 \leq t \leq 6$) where $N(t)$ is measured in millions and t is measured in years, with $t = 1$ corresponding to the beginning of 2005.
 - a) How many Internet users were there at the beginning of 2005? At the beginning of 2006?
 - b) How many Internet users were there expected to be at the beginning of 2010?