Lecture 8

Learning Objectives:

At the end of this class, students should be able to:

- find domain of the function
- find composite function
- plot the graph of the function

8.1 Finding Domain and Range

While finding the domain of a function, we will be concerned to identifying values excluded from the domain. Values excluded from the domain will include any values of independent variable for which the function is not defined. The two exclusions we will focus upon are:

- Any values of the independent variable, which result in the denominator a value of 0 for a quotient function.
- Any values of the independent variable, which result in, no real roots of a radical.

Let us go through the following illustration to understand the situations.

Illustration

Find the domain of the following functions.

i) $f(x)=2x^2+5$ ii) $f(x)=\dfrac{3}{x-5}$ iii) $f(x)=\sqrt{x-5}$

Solution

i) Here $f(x)=2x^2+5$

There are no restrictions on the numbers substituted for *x*, so the domain of this function is the set of all real numbers.

ii) Here, $f(x)=\dfrac{3}{x-5}$

The given function is not defined when the denominator becomes 0.

i. e., $x-5=0$

or, $x=5$

That is, the given function is not defined when x = 5.

Thus, the domain of the given function is the set of all real numbers except 5.

iii) Here, $f(x)=\sqrt{x-5}$

The given function is defined only when the expression under the square root sign is positive or zero.

i.e., $x-5 \geq 0$

or, $x \geq 5$

Thus, the domain of the given function is the set of real numbers which are greater than or equal to 5.

Illustration

Dr. Paul Siple conducted studies testing the effect of wind on the formation of ice at various temperatures and developed the concept of the wind chill, which we hear reported during winter weather reports. Using Siple's original work, if the air temperature is -5°F, then the wind chill,

WC, is a function of the wind speed, s (in mph), and is given by $WC = f(s) = 45.694 + 1.75s - 29.26\sqrt{s}$

a) Based on the formula for $f(s)$ and the physical context of the problem, what is the domain of $f(s)$?
b) Find $f(10)$ and write a sentence that explains its meaning.
c) The working domain of this wind chill function is actually $s \geq 4$. How can you tell that $s = 0$ is not in the working domain, even though it is in the mathematical domain?

Solution

a) Here we have $WC = f(s) = 45.694 + 1.75s - 29.26\sqrt{s}$. Since, we have the term $\sqrt{s}$ in the given function, so the domain of the given function must be $s \geq 0$.
b) Here, $f(10) = 45.694 + 1.75 \times 10 - 29.26\sqrt{10} = -29.33$. This means that if the air temperature is -5°F and there is a 10 mph wind, then the temperature feels like -29.33°F.
c) When we put s = 0, we get $f(0) = 45.694 + 1.75 \times 0 - 29.26\sqrt{0} = 45.694$. But $f(0)$ should equal the air temperature, -5°F.

8.2 Composition of Functions

Given functions $f(u)$ and $g(x)$, the composition $f(g(x))$ is the function of x formed by substituting $u = g(x)$ for u in the formula for $f(u)$.

Illustration

Find the composite function $g(h(x))$ if $g(u) = 3u^2 + 2u - 6,\ h(x) = x + 2$.

Solution

Here, $g(h(x))$

$$= g(x+2)$$
$$= 3(x+2)^2 + 2(x+2) - 6,$$
$$= 3x^2 + 12x + 12 + 2x + 4 - 6$$
$$= 3x^2 + 14x + 10$$

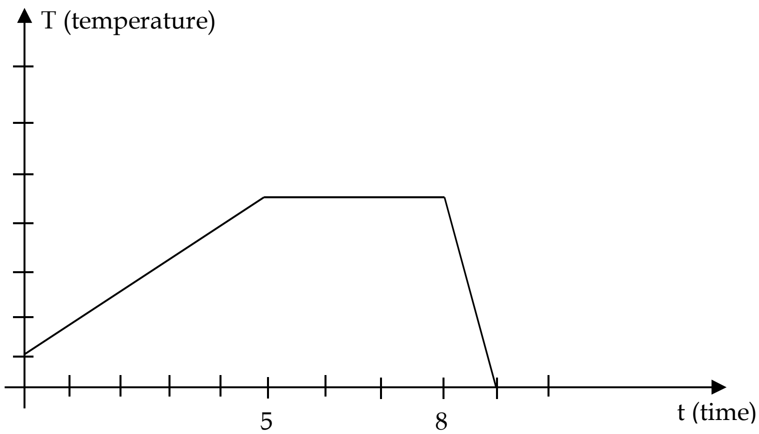
Thus, $g(h(x)) = 3x^2 + 14x + 10$

8.3 The Graph of a Function

Graph is a drawing which depicts a functional relation, e.g., the graph of a function in one variable is (in the plane) the curve which contains those points, and only those points, whose coordinates satisfy the given function.

Thus, the graph of f is the set of points $(x, f(x))$ in a coordinate plane, where x is in the domain of f.

Graphs are often used to describe the variation of physical quantities. For example, a scientist may use the following figure to indicate the temperature T of a certain solution at various time t during an experiment. The sketch shows that the temperature increases gradually for time t = 0 to time t = 5, did not change between t = 5 and t = 8, and then decreased rapidly from t = 8 to t = 9.

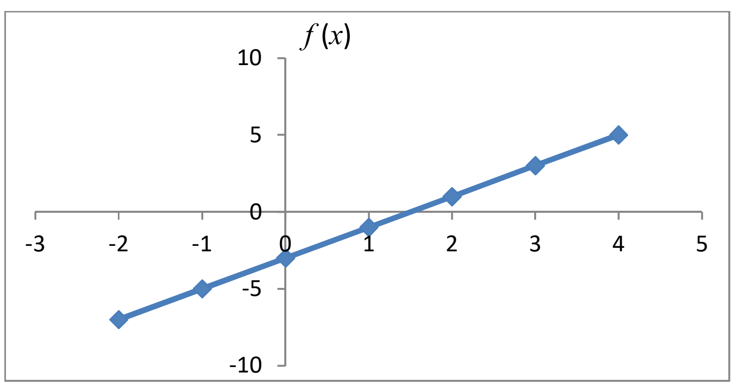


To represent a function geometrically as a graph, we plot values of the independent variable x on the (horizontal) x axis and values of the dependent variable y on the (vertical) y axis.

Illustration
Sketch the graph of the function $f(x) = 2x - 3$.
Solution
Let us list the coordinates $(x, f(x))$ of some points on the graph of f in tabular form, as follows.

x	-2	-1	0	1	2	3	4
$f(x)$	-7	-5	-3	-1	1	3	5

Now plotting these points on a coordinate plane and join them by a straight line, we get the required graph.



Since the given function is defined for all real numbers, therefore, the domain of this function is the set of all real numbers. The range is also the set of real numbers. We can visualize this from the graph.

Vertical Line Test for Function: A curve in the xy-plane is the graph of a function if and only if each vertical line cuts or touches the curves at no more than one point.

Exercise for Reader

1. Find the domain of the following functions.
   i) $f(x) = x + 3$
   ii) $f(x) = \sqrt{x-4}$
   iii) $f(x) = \dfrac{x^2 + 3}{x + 2}$
   iv) $f(x) = \dfrac{2x - 7}{x^3 + x^2 - 6x}$
   v) $f(x) = \sqrt{100 - x}$
2. Let $y = g(u) = \sqrt{5u - 10}$ and $u = h(x) = x^2 - 10x$, determine $g(h(x))$ and $g(h(-1))$.
3. Sketch the graph of the function: $f(x) = x^2 + 1$.