Lecture 7

Learning Objectives:

At the end of this class, students should be able to:

- understand the concept of real number system
- familiar with intervals
- familiar with absolute value
- understand the concept of functions

7.1 Real Numbers

We study about the following numbers before talking about real numbers.

Natural Numbers

The numbers 1, 2, 3, . . . that are used for counting are known as natural numbers. The set of natural numbers is denoted by $\boldsymbol{N}$.
Thus, $\boldsymbol{N} = \{1, 2, 3, \ldots\}$.

Whole Numbers

If we include 0 in the set of natural numbers it becomes the set of whole numbers. The set of whole numbers is denoted by $\boldsymbol{W}$.
Thus, $\boldsymbol{W} = \{0, 1, 2, 3, \ldots\}$.

Integers

All the natural numbers including their negatives and zero are called integers. The set of integers is denoted by $\boldsymbol{Z}$ or $\boldsymbol{I}$.
Thus, $\boldsymbol{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

Rational Numbers

Rational numbers are the numbers that can be written as fractions or ratios. A number which can be expressed in the form p/q, where p and q are integers and $q \neq 0$, is called a rational number. The set of rational numbers is denoted by $\boldsymbol{Q}$.
Thus, $\boldsymbol{Q} = \{p/q: \ p, q \in Z, q \neq 0\}$
For example, the numbers 1/2, 5/7, 9/4 etc. are rational numbers.

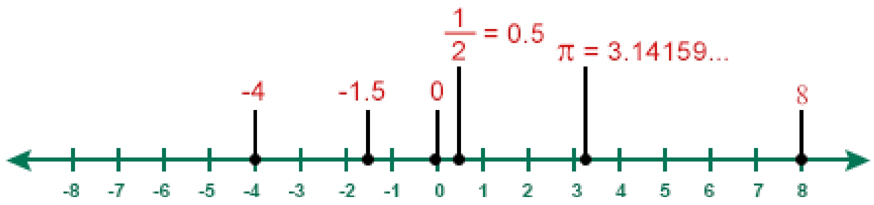
Irrational Numbers

An irrational number is a number that cannot be expressed as a fraction p/q for any integers p and q. Numbers represented by non-terminating and non-repeating decimals are also irrational numbers. Numbers such as $\sqrt{2} = 1.04142\ldots$, $\sqrt{3} = 1.732\ldots$ are examples of irrational numbers. The set of irrational numbers is denoted by Q'.

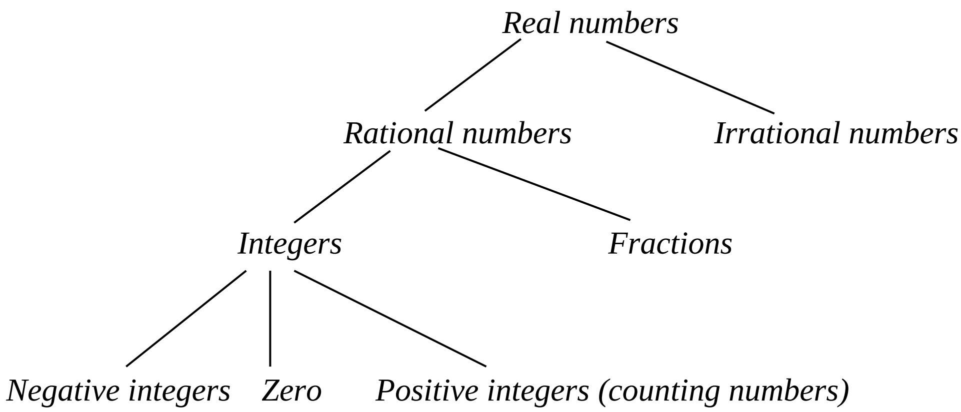
Real Numbers

A collection of rational numbers and irrational numbers constitute the set of real numbers. It is denoted by $\boldsymbol{R}$. So, a real number is either rational or irrational; and $\boldsymbol{R} = Q \cup Q'$.

Numbers can be represented in the number line as follows.



The real number system can be represented by the following tree diagram.



7.2 Intervals

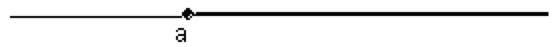
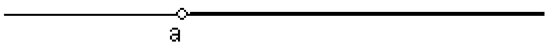
If a < b, the symbol (a, b) denotes all real numbers between a and b. This set is called an *open interval* from a to b, defined by

$$(a, b) = \{x: a < x < b\}$$

The number a and b are called end points of the interval.

If end points a and b are included in the set, the set is written as [a, b] and is called the closed interval from a to b, defied by

$$[a, b] = \{x : a \le x \le b\}$$

Inequality	Geometrical Description	Interval Notation
$a \le x \le b$		$[a, b]$
$a < x < b$		(a, b)
$a \le x < b$		$[a, b)$
$a < x \le b$		$(a, b]$
$a \le x$		$[a, \infty)$
$a < x$		(a, ∞)
$x \le b$		$(-\infty, b]$
$x < b$		$(-\infty, b)$

7.3 Absolute Value

Let x be a real number. The absolute value (or modulus or numerical value) of x, written as $|x|$, is a non-negative number, defined by

$$|x| = \begin{cases} x, & \text{if } x \ge 0 \\ -x, & \text{if } x < 0 \end{cases}$$

For example, $|7| = 7$, $|-4.2| = 4.2$ and $|0| = 0$, therefore, the absolute value of a number is always nonnegative, i.e., $|x| \ge 0$.

Thus, we can write $|x| = \max(x, -x)$

Properties of Absolute Values

1. Let x be any real number, then
   a) $|x| \ge 0$
   b) $|-x| = |x|$
   c) $-x \le |x|$, and $x \le |x|$

2. For any two real numbers x and y, then
   a) $|x - y| = |y - x|$
   b) $|x + y| \le |x| + |y|$
   c) $|x - y| \ge |x| - |y|$
   d) $|x . y| = |x| . |y|$

e) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|} \qquad (y \neq 0)$

3. If a is a positive real number then $|x| < a$ implies $-a < x < a$. Similarly, $|x| \leq a$ implies $-a \leq x \leq a$.
4. If a is a positive real number then $|x| > a$ implies either $x < -a$ or $x > a$. Similarly, $|x| \geq a$ implies either $x \leq -a$ or $x \geq a$.

7.4 Functions

A function is a rule that assigns to each object in a set A exactly one object in a set B. The set A is called the domain of the function, and the set of assigned objects in B is called the range.

Thus, we can say that a function is a correspondence between two sets of elements such that to each element x in the first set, there corresponds one and only one element $y = f\,(x)$ in the second set.

The following are few examples for correspondence.

- To each person, there corresponds a certain age.
- To each medicine in a drugstore, there corresponds a price.
- To each student, there corresponds a grade-point average.

In the functional relation $y = f\,(x)$, the input values are domain values, and the output values are range values. This relation (equation) assigns each domain value x a range value y. The variable x is called an *independent variable* (since values can be independently assigned to x from the domain), and y is called a *dependent variable* (since the value of y depends on the value assigned to x). In general, any variable used as a placeholder for domain values is called an independent variable; any variable that is used as a placeholder for range values is called a dependent variable. For most functions that we will study in this course, the domain and range will be collections of real numbers.

Many formulas in mathematics can be expressed as a function, e.g., the area of a square is a function of its side length. If we denote the length of a side of square by x then its area y can be written as $y = f(x) = x^2$.

Exercise for Reader

1. Use intervals to describe the real numbers satisfying the following inequalities.
   i) $-5 < x < 7$
   ii) $2 \leq x \leq 13$
   iii) $-5 < x \leq 0$
   iv) $x < 7$
2. Rewrite the following by using absolute sign.
   i) $-2 \leq x \leq 8$
   ii) $-4 < x < -2$
3. Rewrite the following without using absolute value sign.
   i) $|2x - 1| < 4$
   ii) $|x + 3| \leq 6$