Lecture 4

Learning Objectives
At the end of this class, students should be able to:

- solve quadratic equations
- solve quadratic inequalities

4.1 Quadratic Equations

An equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$ is called quadratic equation; where a, b, and c are constants. For example, the equations $x^2 - 4x + 5 = 0$ and $3x^2 - 9 = 0$ are quadratic equations.

One of the methods for solving quadratic equation is factoring technique. Solution by factoring is based on the following property of the real numbers. For real numbers a and b, $a.b = 0$ if and only if $a = 0$ or $b = 0$ or both.

Illustration 1

Solve the equation: $x^2 + 12x + 36 = 0$.

Solution

We have $x^2 + 12x + 36 = 0$

or $(x)^2 + 2 \times x \times 6 + (6)^2 = 0$

or $(x+6)^2 = 0$

or $x + 6 = 0$

Thus, x = 6 is the solution of given equation.

Illustration 2

Solve the equation: $x^2 + x - 56 = 0$

Solution

We have $x^2 + x - 56 = 0$

or $x^2 + (8-7)x - 56 = 0$

or $x^2 + 8x - 7x - 56 = 0$

or $x(x+8) - 7(x+8) = 0$

or $(x+8)(x-7) = 0$

or $x = -8, 7$

Thus, the solutions of the given equation are -8 and 7.

Note: The solution of the quadratic equation of the form $x^2 = d$ (no *x*-term) is $x = \pm\sqrt{d}$.

Illustration 3

Determine the solution of the equation: $(2x-1)^2 = 4$.

Solution

We have $(2x-1)^2 = 4$. Then

$(2x-1) = \pm\sqrt{4}$

i.e., $(2x-1) = \sqrt{4}$ or $(2x-1) = -\sqrt{4}$

i.e., $2x-1 = 2$ or $2x-1 = -2$

i.e., $x = 3/2$ or $x = -1/2$

Thus, the solutions are: *x* = 3/2 and *x* = -1/2.

4.2 Quadratic Formula

The general form of the equation is given by,

$$ax^2 + bx + c = 0, \text{ where } a \neq 0$$

Let us identify its solutions by completing the square:

First, dividing both sides by a and putting the constant term in the right side, we get

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Adding $\frac{b^2}{4a^2}$ to both sides of the equation, we get

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

or $$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$$

Extracting the square root, we get

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

or $$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Thus, we get two solutions $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$ of the quadratic equation $ax^2 + bx + c = 0$.

Thus, two solutions (roots) of a quadratic equation $ax^2 + bx + c = 0$ can be found directly by applying the formula: $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ and this is known as quadratic formula.

Illustration 4

Solve the quadratic equation: $x^2 + 5x + 4 = 0$

Solution

Comparing the given equation with $ax^2 + bx + c = 0$, we get a = 1, b = 5 and c = 4. Thus,

$$x = \frac{-5 \pm \sqrt{5^2 - 4 \times 1 \times 4}}{2 \times 1}$$

$$= \frac{-5 \pm \sqrt{9}}{2}$$

$$= \frac{-5 \pm 3}{2}$$

Thus $x = \frac{-5+3}{2}$ or $x = \frac{-5-3}{2}$

i.e. $x = -1$ or $x = -4$

Thus, the solutions of the given equation are -1 and -4.

4.3 Quadratic Inequalities

When we have quadratic inequality, we can solve it by using factorization. The following illustrations provide solution procedure.

Illustration 5

Solve the inequality: $x^2 + 3x + 2 \geq 0$.

Solution

We have $x^2 + 3x + 2 \geq 0$

or $(x + 1)(x + 2) \geq 0$

The following attributes of the two factors on the left-hand side will result in the inequality being satisfied.

Condition	Factor		Product
	(x + 1)	(x + 2)	
I	= 0	Any value	0
II	Any value	= 0	0
III	> 0	> 0	> 0
IV	< 0	< 0	> 0

Condition I: $x + 1 = 0$ when $x = -1$

Condition II: $x + 2 = 0$ when $x = -2$

Condition III: $x + 1 > 0$ and $x + 2 > 0$

when $x > -1$ and $x > -2$

Condition IV: $x + 1 < 0$ and $x + 2 < 0$

when $x < -1$ and $x < -2$

From conditions I and II, we get $x = -1$ and $x = -2$. From condition III, we get $x > -1$. From condition III, we get $x < -2$.

Thus, from these conditions, the required solution of the given inequality is $x \leq -2$ or $x \geq -1$.

Illustration 6

Solve the inequality: $x^2 - 4 \leq 0$.

Solution

We have $x^2 - 4 \leq 0$

or $(x - 2)(x + 2) \leq 0$

The following attributes of the two factors on the left-hand side will result in the inequality being satisfied.

Condition	Factor		Product
	(x – 2)	(x + 2)	
I	= 0	Any value	0
II	Any value	= 0	0
III	> 0	< 0	< 0
IV	< 0	> 0	< 0

Condition I: $x - 2 = 0$ when $x = 2$

Condition II: $x + 2 = 0$ when $x = -2$

Condition III: $x - 2 > 0$ and $x + 2 < 0$

when $x > 2$ and $x < -2$

Condition IV: $x - 2 < 0$ and $x + 2 > 0$

when $x < 2$ and $x > -2$

From conditions I and II, we get $x = 2$ and $x = -2$. Condition III does not provide any solution, while condition IV provides $-2 < x < 2$.

Combining all these results, we get $2 \leq x \leq -2$. Thus, the solution to the given inequality is $2 \leq x \leq -2$.

Exercise for Reader

1. Solve the following quadratic equations.

   a) $x^2 - 7x + 6 = 0$

   b) $2x^2 + x - 4 = 0$

   c) $15x^2 - 28 = x$

   d) $x - \frac{1}{x} = 3, \qquad x \neq 0$

2. A two-digit number is such that the product of digits is 20. When 9 is added to the number then the digits interchange their places. Find the number.

3. Solve the following inequalities.

   a) $(x - 2)(x + 5) \leq 0$

   b) $x^2 - 16 \geq 0$