

THE WKB METHOD

(Liouville 1837, Green 1837, Horn 1899, Rayleigh 1912, Gans 1915, Jeffrey 1923, Wentzel 1926, Kramers 1926, Brillouin 1926, Langer 1931, Olver 1961, Meyer 1973. Notice how it's not named after the scientists who discovered it. The WKB method achieved prominence in the 20th century in its use in semiclassical analysis of quantum problems, among other areas.)

One example of a singular perturbation problem that does **not** have boundary layers is

$$\epsilon^2 y'' + y = 0.$$

It has **oscillatory** solutions and is typical of many problems arising from wave propagation, with $\epsilon =$ wavelength/size of region. So, for high-frequency propagation, ϵ is small and we need a way to deal asymptotically with such problems. The WKB method is such a method for linear wave propagation problems, and is illustrated by the equation

$$\epsilon^2 y'' + q(x)y = 0, \tag{2}$$

with $q(x) \neq 0$ in the region of interest.

Let us first see what happens if we try to solve the problem by multiple scales. Let $\epsilon X = x$ to give

$$\frac{d^2 y}{dX^2} + q(\epsilon X)y = 0.$$

Thus we have an oscillator with a slowly varying frequency. We might be tempted to write $y = y(x, X)$, giving

$$\frac{\partial^2 y}{\partial X^2} + 2\epsilon \frac{d^2 y}{\partial x \partial X} + \epsilon^2 \frac{\partial^2 y}{\partial x^2} + q(x)y = 0. \tag{3}$$

Expanding $y \sim y_0 + \epsilon y_1 + \dots$ gives at leading order

$$\frac{\partial^2 y_0}{\partial X^2} + q(x)y_0 = 0.$$

Hence

$$y_0 = A(x) \cos(q(x)^{1/2} X + \theta(x)),$$

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where $A(x)$ and $\theta(x)$ are arbitrary functions of x , to be determined by secularity conditions at next order. Equating coefficients of ϵ^1 in (3) gives

$$\frac{\partial^2 y_1}{\partial X^2} + 2 \frac{d^2 y_0}{\partial x \partial X} + q(x)y_1 = 0,$$

i.e.

$$\begin{aligned} \frac{\partial^2 y_1}{\partial X^2} + q(x)y_1 &= 2 \frac{\partial}{\partial x} \left(A(x)q(x)^{1/2} \sin(q(x)^{1/2}X + \theta(x)) \right) \\ &= 2 \frac{d}{dx} \left(Aq^{1/2} \right) \sin(q^{1/2}X + \theta) - 2Aq^{1/2} \left(X \frac{dq^{1/2}}{dx} + \frac{d\theta}{dx} \right) \cos(q^{1/2}X + \theta). \end{aligned}$$

The secularity condition says that there can be no multiple of $\cos(q^{1/2}X + \theta)$ or $\sin(q^{1/2}X + \theta)$ on the right-hand side. Hence

$$\frac{d}{dx} \left(Aq^{1/2} \right) = 0, \quad X \frac{dq^{1/2}}{dx} + \frac{d\theta}{dx} = 0.$$

Here we see a problem though. The second secularity condition contains the fast scale X , and so cannot be satisfied since θ is a function of the slow scale x only. This will happen whenever the frequency of the fast oscillation depends on the slow scale.

Let us now return to (2). Instead of using multiple scales, we assume a WKB asymptotic expansion for y of the form

$$y = e^{i\phi(x)/\epsilon} A(x, \epsilon),$$

with

$$A(x, \epsilon) \sim \sum_{n=0}^{\infty} A_n(x) \epsilon^n.$$

This gives

$$y'' \sim e^{i\phi/\epsilon} \left(-\frac{(\phi')^2 A}{\epsilon^2} + \frac{2i\phi' A'}{\epsilon} + \frac{i\phi'' A}{\epsilon} + A'' \right),$$

so that substituting the expansions into the equation gives at leading order ($O(\epsilon^0)$):

$$\phi'(x)^2 = q_0(x).$$

Hence

$$\phi = \pm \sqrt{q_0(x)}.$$

At order ϵ^1 we find

$$2\phi' A'_0 + \phi'' A_0 = 0,$$

while at order ϵ^{n+1} for $n \geq 1$ we find

$$A''_{n-1} + 2i\phi' A'_n + i\phi'' A_n = 0.$$

These are successive first-order **linear** equations for A_i . The first is

$$\frac{2A'_0}{A_0} + \frac{\phi''}{\phi'} = 0,$$

which we can integrate to

$$2 \log A_0 + \log \phi' = \text{const.},$$

i.e.

$$A_0 = \frac{\alpha_0}{q_0(x)^{1/4}},$$

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for some constant α_0 . In a wave propagation problem this $A_0(x)$ gives the amplitude, and this equation corresponds to energy conservation.

The equation for A_n can be solved using an integrating factor giving

$$2i(\phi')^{1/2} \left((\phi')^{1/2} A_n \right)' = -A_{n-1}''$$

i.e.

$$A_n = \frac{i}{2(\phi')^{1/2}} \int \frac{A_{n-1}''}{(\phi')^{1/2}} dx;$$

the right-hand side is known.

Example The Legendre polynomial $P_n(x)$. If we let $y(\theta) = \sqrt{\sin \theta} P_n(\cos \theta)$ for $0 < \theta < \pi$ then the equation satisfied by y is

$$y'' + \left(n^2 + n + \frac{1}{4} + \frac{1}{4 \sin^2 \theta} \right) y = 0.$$

Let $\epsilon = 1/(n + 1/2)$. Then

$$\epsilon^2 y'' + \left(1 + \frac{\epsilon^2}{4 \sin^2 \theta} \right) y = 0.$$

With the WKB ansatz $y = Ae^{i\phi/\epsilon}$ at leading order (ϵ^0)

$$(\phi')^2 = 1, \quad \phi' = \pm 1, \quad \phi = \pm \theta.$$

At order ϵ^1 we have

$$2\phi' A_0' + \phi'' A_0 = 0,$$

i.e.

$$A_0' = 0, \quad A_0 = \alpha_0.$$

At order ϵ^2 we have

$$A_0'' + 2i\phi' A_1' + i\phi'' A_1 + \frac{1}{4 \sin^2 \theta} A_0 = 0,$$

i.e.

$$2iA_1' = \mp \frac{\alpha_0}{4 \sin^2 \theta}$$

so that

$$A_1 = \mp \frac{i\alpha_0 \cot \theta}{8}.$$

Thus

$$\sqrt{\sin \theta} P_n(\cos \theta) \sim \hat{\alpha}_0 \left(1 - \frac{i \cot \theta}{8(n + 1/2)} \dots \right) e^{i(n+1/2)\theta} + \hat{\beta}_0 \left(1 + \frac{\cot \theta}{8(n + 1/2)} \dots \right) e^{-i(n+1/2)\theta}$$

as $n \rightarrow \infty$.

Example. An eigenvalue problem Find the large eigenvalues $\lambda \gg 1$ of the Sturm-Liouville problem

$$y'' + \lambda p(x)y = 0 \quad \text{for } 0 < x < 1, \quad \text{with } y(0) = 0, \quad y(1) = 0,$$

where $p(x) > 0$ for $0 \leq x \leq 1$. Put $\lambda = \epsilon^{-2}$, where for $\lambda \gg 1$ we require $0 < \epsilon \ll 1$, so that

$$\epsilon^2 y'' + p(x)y = 0 \quad \text{for } 0 < x < 1, \quad \text{with } y(0) = 0, \quad y(1) = 0.$$

Then with the WKB approximation $y \sim Ae^{i\phi/\epsilon}$ as $\epsilon \rightarrow 0^+$, we have

$$\phi' = \pm p^{1/2}, \quad A_0 \propto \frac{1}{(\phi')^{1/2}}.$$

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If we fix $\phi(x) = + \int_0^x p(s)^{1/2} ds$ and $A_0(x) = p(x)^{-1/4}$, then the two independent solutions are given by

$$y_+ \sim A_0 e^{i\phi/\epsilon}, \quad y_- \sim A_0 e^{-i\phi/\epsilon}.$$

Hence, at leading order the general solution may be written in the form

$$y(x) \sim \alpha A_0(x) \cos\left(\frac{\phi(x)}{\epsilon}\right) + \beta A_0(x) \sin\left(\frac{\phi(x)}{\epsilon}\right)$$

as $\epsilon \rightarrow 0^+$, where α and β are arbitrary real constants. The boundary condition $y(0) = 0$ requires $\alpha = 0$, so that the boundary condition $y(1) = 0$ is satisfied at leading order only if

$$\beta A_0(1) \sin\left(\frac{\phi(1)}{\epsilon}\right) = o(1) \quad \text{as } \epsilon \rightarrow 0^+.$$

Since $A_0(1) > 0$ and $\beta \neq 0$ for a nontrivial solution, we require

$$\sin\left(\frac{\phi(1)}{\epsilon}\right) = o(1) \quad \text{as } \epsilon \rightarrow 0^+,$$

i.e.

$$\frac{\phi(1)}{\epsilon} \sim n\pi \quad \text{as } \epsilon \rightarrow 0^+, \quad n \rightarrow \infty \text{ with } n \in \mathbb{N}.$$

The eigenvalues are therefore given approximately by

$$\epsilon_n \sim \frac{\phi(1)}{n\pi} = \frac{\int_0^1 \sqrt{p(x)} dx}{n\pi}$$

or

$$\lambda_n \sim \left(\frac{n\pi}{\int_0^1 \sqrt{p(x)} dx} \right)^2$$

as $n \rightarrow \infty$ with $n \in \mathbb{N}$.

Example. Turning points Find the large eigenvalues $\lambda \gg 1$ of the harmonic oscillator

$$-y'' + x^2 y = \lambda y \quad \text{for } -\infty < x < \infty, \quad \text{with } y \rightarrow 0 \text{ as } |x| \rightarrow \infty.$$

For $\lambda \gg 1$ again let $\epsilon = 1/\lambda$ and rescale $x = \epsilon^{-1/2} \bar{x}$ to give (dropping the bars)

$$\epsilon^2 y'' + (1 - x^2) y = 0 \quad \text{for } -\infty < x < \infty, \quad \text{with } y \rightarrow 0 \text{ as } |x| \rightarrow \infty.$$

Using the WKB *ansatz* we find

$$\phi' = \pm \sqrt{1 - x^2}, \quad A_0 \propto \frac{1}{(1 - x^2)^{1/4}}.$$

Hence, the general solution has the expansion

$$y \sim \frac{\alpha_0}{(1 - x^2)^{1/4}} e^{i\phi(x)/\epsilon} + \frac{\beta_0}{(1 - x^2)^{1/4}} e^{-i\phi(x)/\epsilon} \quad \text{as } \epsilon \rightarrow 0^+, \quad (4)$$

where α_0 and β_0 are arbitrary complex constants and we have fixed $\phi(x) = \int_0^x (1 - s^2)^{1/2} ds$, but this approximation is only good for $|x| < 1$. When x is close to ± 1 , $(1 - x^2)$ is small, and the WKB approximation breaks down. At these places $\phi' = 0$ (so they are known as turning points), and hence $A_0 = \infty$ (which indicates the breakdown). We must use a different expansion in the vicinity of each turning point (an “inner expansion”) and match it with this “outer expansion”.

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Before we do the inner expansion, let us continue with the outer expansion for $|x| > 1$. Then we can still use WKB, and we find that, in $x > 1$ say

$$y \sim \frac{\alpha_1}{(x^2 - 1)^{1/4}} e^{-\frac{1}{\epsilon} \int_1^x (s^2 - 1)^{1/2} ds} + \frac{\beta_1}{(s^2 - 1)^{1/4}} e^{\frac{1}{\epsilon} \int_1^x (s^2 - 1)^{1/2} ds}, \quad (5)$$

where α_1 and β_1 are arbitrary real constants. Now we can apply the boundary condition at $x = +\infty$ to give $\beta_1 = 0$. The inner region near $x = 1$ will allow us to connect the coefficients α_0 and β_0 to α_1 and β_1 . This will give us one condition on α_0 and β_0 . The inner region near $x = -1$ will give us another.

Locally near $x = 1$ we rescale $x = 1 + \epsilon^{2/3} \hat{x}$, $y = \epsilon^{-1/6} \hat{y}(\hat{x})$ to give at leading order

$$\frac{d^2 \hat{y}}{d\hat{x}^2} - 2\hat{x}\hat{y} = 0.$$

This is just the Airy equation. We want a solution which matches with (5) as $\hat{x} \rightarrow \infty$. This solution is

$$\hat{y} = CAi\left(2^{1/3}\hat{x}\right),$$

where Ai is the Airy function. It can be shown that as $\hat{x} \rightarrow \infty$

$$\epsilon^{-1/6} \hat{y}(\hat{x}) = \epsilon^{-1/6} CAi\left(2^{1/3}\hat{x}\right) \sim \frac{C}{2^{13/12} \sqrt{\pi} (\epsilon^{2/3} \hat{x})^{1/4}} e^{-2\sqrt{2}\hat{x}^{3/2}/3},$$

while the inner limit of (5) is

$$\frac{\alpha_1}{(2\epsilon^{2/3}\hat{x})^{1/4}} e^{-2\sqrt{2}\hat{x}^{3/2}/3}.$$

Hence, matching the two expansions gives

$$C = \frac{2^{13/12} \sqrt{\pi} \alpha_1}{2^{1/4}}.$$

Now as $\hat{x} \rightarrow -\infty$, it can be shown that

$$\epsilon^{-1/6} \hat{y}(\hat{x}) = \epsilon^{-1/6} CAi\left(-2^{1/3}\hat{x}\right) \sim -\frac{C e^{-i\pi/4}}{2^{13/12} \sqrt{\pi}} \left(\frac{1}{(\epsilon^{2/3}\hat{x})^{1/4}} e^{-2\sqrt{2}i\hat{x}^{3/2}/3} - \frac{i}{(\epsilon^{2/3}\hat{x})^{1/4}} e^{2\sqrt{2}i\hat{x}^{3/2}/3} \right),$$

while the inner limit of (4) is

$$\frac{\alpha_0}{(2\epsilon^{2/3}\hat{x})^{1/4}} e^{-\frac{i}{\epsilon}\phi(1)} e^{-2\sqrt{2}i\hat{x}^{3/2}/3} + \frac{\beta_0}{(2\epsilon^{2/3}\hat{x})^{1/4}} e^{\frac{i}{\epsilon}\phi(1)} e^{2\sqrt{2}i\hat{x}^{3/2}/3}.$$

Matching the two expansions requires

$$\alpha_0 e^{-\frac{i}{\epsilon}\phi(1)} \sim -\frac{C 2^{1/4} e^{-i\pi/4}}{2^{13/12} \sqrt{\pi}}, \quad \beta_0 e^{\frac{i}{\epsilon}\phi(1)} \sim \frac{C i 2^{1/4} e^{-i\pi/4}}{2^{13/12} \sqrt{\pi}}$$

as $\epsilon \rightarrow 0^+$. For nontrivial solution (*i.e.* $C \neq 0$), we therefore require

$$\alpha_0 e^{-\frac{i}{\epsilon}\phi(1)} \sim i\beta_0 e^{\frac{i}{\epsilon}\phi(1)}$$

as $\epsilon \rightarrow 0^+$. Similarly, through a local analysis at $x = -1$ we find

$$\alpha_0 e^{-\frac{i}{\epsilon}\phi(-1)} \sim -i\beta_0 e^{\frac{i}{\epsilon}\phi(-1)}$$

as $\epsilon \rightarrow 0^+$. Hence, for a nonzero solution α_0, β_0 to exist we need

$$\frac{e^{-\frac{i}{\epsilon}\phi(1)}}{e^{-\frac{i}{\epsilon}\phi(-1)}} \sim \frac{ie^{\frac{i}{\epsilon}\phi(1)}}{-ie^{\frac{i}{\epsilon}\phi(-1)}},$$

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giving

$$e^{-\frac{2i}{\epsilon}(\phi(1)-\phi(-1))+i\pi} \sim 1,$$

so that

$$\frac{2(\phi(1) - \phi(-1))}{\epsilon} \sim (2n + 1)\pi \quad \text{as } \epsilon \rightarrow 0^+, \quad n \rightarrow \infty \text{ with } n \in \mathbb{N}.$$

Hence, the eigenvalues are given approximately by

$$\epsilon_n \sim \frac{\phi(1) - \phi(-1)}{(n + 1/2)\pi} = \frac{\int_{-1}^1 \sqrt{1-x^2} dx}{(n + 1/2)\pi} = \frac{1}{2n + 1}$$

or

$$\lambda_n = \frac{1}{\epsilon_n} \sim 2n + 1$$

as $n \rightarrow \infty$ with $n \in \mathbb{N}$. In fact these are **exact** in this case. The exact solutions are

$$y_n = e^{-x^2/2} H_n(x), \quad \lambda_n = 2n + 1.$$