MULTIPLE-SCALE PERTURBATION THEORY

Of all asymptotic techniques, this is the one which is the most like a "black art". Problems characterised by having two processes, each with their own scales, acting simultaneously. Rapidly varying phase, slowly varying amplitude; modulated waves. Contrast with matched asymptotic expansions, where the two processes with different scales are acting in different regions.

Example: back to van der Pol oscillator

$$\ddot{x} + \epsilon \dot{x}(x^2 - 1) + x = 0.$$

Last time we looked at relaxation oscillations for large ϵ (called μ then). Here will will study with small $\epsilon > 0$ the initial value problem with initial conditions

$$x = 1, \quad \dot{x} = 0 \quad \text{at } t = 0.$$

Treating the problem as a regular perturbation expansion in ϵ gives

$$x(t, \epsilon) \sim \cos t + \epsilon \left[\frac{3}{8}(t \cos t - \sin t) - \frac{1}{32}(\sin 3t - 3 \sin t) \right] + \cdots .$$

This expansion is valid for fixed t as $\epsilon \to 0$, but breaks down when $t \geq O(\epsilon^{-1})$, because of the resonant terms. When the second term in an expansion becomes as big as the first it is an indication that the expansion is breaking down.

The problem is that the damping term only changes the amplitude by an order one amount over a timescale of order ϵ^{-1}, by a slow accumulation of small effects. Thus the two processes on the two time scales are fast oscillation and slow damping.

We try to capture the behaviour on both these timescales by introducing **two** time variables:

$$\begin{aligned} \tau &= t \quad &\text{— the fast time of the oscillation,} \\ T &= \epsilon t \quad &\text{— the slow time of the amplitude drift.} \end{aligned}$$

We look for a solution of the form $x(t; \epsilon) = x(\tau, T; \epsilon)$ treating the variables τ and T as **independent**. We have

$$\frac{d}{dt} = \frac{d\tau}{dt}\frac{\partial}{\partial \tau} + \frac{dT}{dt}\frac{\partial}{\partial T} = \frac{\partial}{\partial \tau} + \epsilon \frac{\partial}{\partial T},$$

so that

$$\ddot{x} = x_{\tau\tau} + 2\epsilon x_{\tau T} + \epsilon^2 x_{TT}.$$

Then we expand

$$x(\tau, T; \epsilon) \sim x_0(\tau, T) + \epsilon x_1(\tau, T) + \cdots \quad \text{as } \epsilon \to 0.$$

At ϵ^0 we find

$$x_{0\tau\tau} + x_0 = 0 \quad \text{in } t \geq 0,$$

with

$$x_0 = 1, \quad x_{0\tau} = 0 \quad \text{at } t = 0.$$

Hence

$$x_0 = R(T) \cos(\tau + \theta(T)).$$

Thus the amplitude and phase are constant as far as the fast timescale τ is concerned, but vary over the slow timescale T. Applying the initial conditions we require

$$R(0) = 1, \qquad \theta(0) = 0.$$

Apart from these conditions R and θ are arbitrary at present. Proceeding to order ϵ^1:

$$\begin{aligned} x_{1\tau\tau} + x_1 &= -x_{0\tau}(x_0^2 - 1) - 2x_{0\tau T} \qquad \text{in } t \geq 0 \\ &= 2R\theta_T \cos(\tau + \theta) + \left(2R_T + \frac{R^3}{4} - R\right) \sin(\tau + \theta) + \frac{R^3}{4} \sin 3(\tau + \theta). \end{aligned}$$

The initial conditions are

$$x_1 = 0, \qquad x_{1\tau} = -x_{oT} = -R_T \quad \text{at } t = 0.$$

Now, the $\sin 3(\tau+\theta)$ term is OK, but the $\sin(\tau+\theta)$ and $\cos(\tau+\theta)$ terms are resonant, and will give a response of the form $t\sin(\tau+\theta)$ and $t\cos(\tau+\theta)$. Thus the expansion will cease to be asymptotic again when $t = O(\epsilon^{-1})$. To keep the expansion asymptotic, we use the freedom we have in R and θ to eliminate these resonant terms (the so-called secularity or integrability or solvability condition of Poincaré), giving

$$\theta_T = 0, \qquad R_T = \frac{R(4-R^2)}{8}.$$

Using the initial conditions we therefore have

$$\theta = 0, \qquad R = \frac{2}{(1+3e^{-T})^{1/2}}.$$

Thus the amplitude of the oscillator drifts towards the value 2, which we found was a limit cycle. Thus, in particular, we have shown that the limit cycle is stable.

If we are interested in the correction x_1 we can now calculate it as

$$x_1 = -\frac{1}{32}R^3 \sin 3\tau + S(T)\sin(\tau + \phi(T)),$$

with new amplitude and phase functions S and ϕ. These will be determined by a secularity condition on x_2, etc.

At higher orders we would find that a resonant forcing is impossible to avoid. In fact this is the case here in solving for x_1: we cannot avoid resonance in x_2. This can be avoided by introducing an additional slow timescale $T_2 = \epsilon^2 t$.

A simple example which illustrates the need for such a super slow time scale is the damped linear oscillator

$$\ddot{x} + 2\epsilon\dot{x} + x = 0$$

with solution

$$x = e^{-\epsilon t}\cos\left(\sqrt{1-\epsilon^2}\,t\right).$$

The amplitude drifts on the timescale ϵ^{-1}, while the phase drifts on the timescale ϵ^{-2}. In general, if we want the solution correct to $O(\epsilon^k)$ for times of $O(\epsilon^{k-n})$ then we need a hierarchy of n slow timescales.

Example: the van der Pol oscillator again

$$\ddot{x} + \epsilon\dot{x}(x^2-1) + x = 0.$$

In practice we often work directly with the variable t to save introducing the variable τ and make use of the complex representation of trigonometric functions to simplify the algebra. Thus, in seeking a multiple scales solution we begin by substituting

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \epsilon\frac{\partial}{\partial T}$$

to obtain

$$x_{tt} + 2\epsilon x_{tT} + \epsilon^2 x_{TT} + \epsilon\dot{x}(x^2-1) + x = 0.$$

Expanding

$$x \sim x_0(t,T) + \epsilon x_1(t,T) + \cdots \quad \text{as } \epsilon \to 0,$$

we obtain at leading order

$$x_{0tt} + x_0 = 0.$$

The general solution of this PDE has the complex representation

$$x_0 = \frac{1}{2}\left(A(T)e^{it} + \overline{A}(T)e^{-it}\right),$$

where A is an arbitrary complex function of T, $\overline{A}$ is the complex conjugate of A and the pre-factor of $1/2$ has been introduced so that $|A(T)|$ is the slowly-varying amplitude and $\arg(A(T))$ is the slowly-varying phase, *e.g.* if $A(T) = R(T)e^{i\Theta(T)}$, where $R(T) \geq 0$, then $x_0 \equiv R(t)\cos(it + \Theta(T))$.

At $O(\epsilon^1)$, we obtain

$$\begin{aligned} x_{1tt} + x_1 &= -x_{0t}(x_0^2 - 1) - 2x_{0tT} \\ &= -\frac{1}{2}\left(iAe^{it} - i\overline{A}e^{-it}\right)\left(\frac{1}{4}(Ae^{it} + \overline{A}e^{-it})^2 - 1\right) - \left(iA_Te^{it} - i\overline{A}_Te^{-it}\right), \\ &= -i\left(\frac{dA}{dT} - \frac{A(4-|A|^2)}{8}\right)e^{it} + \text{ complex conjugate term } + \text{ non-secular terms.} \end{aligned}$$

Secular terms proportional to $e^{\pm it}$ are suppressed only if $A(T)$ satisfies the ODE

$$A_T = \frac{A(4-|A|^2)}{8}.$$

Substituting $A(T) = R(T)e^{i\Theta(T)}$, where $R(T) \geq 0$, we recover the ODEs

$$\Theta_T = 0, \quad R_T = \frac{R(4-R^2)}{8}.$$