TURNING POINTS

- $-y'' + x^2 y = \lambda y$ for $x > 0$, with $y'(0) = 0$, $y(\infty) = 0$.

- To find eigenvalues $\lambda \gg 1$, write ODE in "WKB form" by scaling $\lambda = \varepsilon^{-1}$, $x = \varepsilon^{-1/2}\bar{x}$, with $0 < \varepsilon \ll 1$, to give (dropping bars):

$$\varepsilon^2 y'' + (1 - x^2) y = 0 \text{ for } x > 0, \text{ with } y'(0) = 0, \; y(\infty) = 0.$$

- Try WKB ansatz: $y \sim e^{i\phi(x)/\varepsilon} \sum_{n=0}^{\infty} A_n(x)\varepsilon^n$ as $\varepsilon \to 0^+$

$$\Rightarrow \quad O(\varepsilon^0) \; : \; (\phi')^2 = 1 - x^2$$

$$O(\varepsilon^1) \; : \; (A_0^2 \phi')' = 0$$

- Hence,

$$|x| < 1 \quad \Rightarrow \quad \phi' = \pm(1 - x^2)^{1/2}, \quad A_0 \propto (1 - x^2)^{-1/4}$$

$$|x| > 1 \quad \Rightarrow \quad \phi' = \pm i(1 - x^2)^{1/2}, \quad A_0 \propto (x^2 - 1)^{-1/4}$$

- Hence, WKB works for $|x| - 1 = \text{ord}(1)$ as $\varepsilon \to 0^+$, but as $|x| \to 1$, $|A_0(x)| \to \infty$ and method breaks down.

- This is because $\phi'(\pm 1) = 0$ and $A_0^2 \phi' = \text{const}$.

- Points at which $\phi' = 0$ called turning points (here $x = \pm 1$).

- Resolve breakdown using method of matched asymptotic expansions.

RH outer $x > 1$, $x - 1 = \mathrm{ord}(1)$ as $\varepsilon \to 0^+$

- $y(\infty) = 0 \Rightarrow$ need to eliminate growing solution, leaving

$$y \sim C_1 A_0^R(x) \exp\left(-\frac{\phi^R(x)}{\varepsilon}\right) \quad \text{as } \varepsilon \to 0^+,$$

where $C_1 \in \mathbb{R}$ and we fix

$$A_0^R(x) = (x^2 - 1)^{-1/4} \sim (2(x-1))^{-1/4} \quad \text{as } x \to 1^+,$$

and

$$\phi^R(x) = \int_1^x (s^2 - 1)^{1/2}\,ds \sim \frac{2^{3/2}}{3}(x-1)^{3/2} \quad \text{as } x \to 1^+.$$

LH outer $0 < x < 1$, $1 - x = \mathrm{ord}(1)$ as $\varepsilon \to 0^+$

- Now both $\phi' = \pm(1 - x^2)^{1/2}$ are admissible, but $y'(0) = 0$, so

$$y \sim C_2 A_0^L(x) \cos\left(\frac{\phi^L(x)}{\varepsilon}\right) \quad \text{as } \varepsilon \to 0^+.$$

where $C_2 \in \mathbb{R}$ and we fix

$$A_0^L(x) = (1 - x^2)^{-1/4} \sim (2(1-x))^{-1/4} \quad \text{as } x \to 1^-,$$

and

$$\phi^L(x) = \int_0^x (1 - s^2)^{1/2}\,ds \sim \frac{\pi}{4} - \frac{2^{3/2}}{3}(1-x)^{3/2} \quad \text{as } x \to 1^-.$$

Inner region near $x = 1$

- Scale $x = 1 + \delta(\varepsilon)X$, $y = \delta(\varepsilon)^{-1/4}Y(X)$, with $\delta(\varepsilon) \to 0$ and $X = \text{ord}(1)$ as $\varepsilon \to 0^+$

$$\Rightarrow \quad \frac{\varepsilon^2}{\delta^2}\frac{d^2Y}{dX^2} - 2\delta XY - \delta^2 X^2 Y = 0$$

- Only possible dominant balance for $\delta(\varepsilon) \ll 1$ is between 1^{st} and 2^{nd} terms, so let $\frac{\varepsilon^2}{\delta^2} = 2\delta$ or $\delta = \frac{\varepsilon^{2/3}}{2^{1/3}}$.
- Try $Y \sim Y_0(X) + \varepsilon^{2/3}Y_1(X) + \cdots$ as $\varepsilon \to 0^+$

$$\Rightarrow \quad O(\varepsilon^0): \quad \frac{d^2Y_0}{dX^2} - XY_0 = 0 \quad \text{(Airy's equation)}$$

- General solution is $Y_0 = C_3\,\mathrm{Ai}(X) + C_4\,\mathrm{Bi}(X)$ $\quad (C_3, C_4 \in \mathbb{R})$

Airy functions

- $\mathrm{Ai}(X) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + Xt\right) dt.$
- $\mathrm{Ai}(X) \sim \frac{1}{2\sqrt{\pi}X^{1/4}} \exp\left(-\frac{2}{3}X^{3/2}\right)$ as $X \to \infty$.
- $\mathrm{Ai}(X) \sim \frac{1}{\sqrt{\pi}(-X)^{1/4}} \sin\left(\frac{2}{3}(-X)^{3/2} + \frac{\pi}{4}\right)$ as $X \to -\infty$.
- $\mathrm{Bi}(X) = \frac{1}{\pi}\int_0^\infty \exp\left(-\frac{t^3}{3} + Xt\right) + \sin\left(\frac{t^3}{3} + Xt\right) dt.$
- $\mathrm{Bi}(X) \sim \frac{1}{\sqrt{\pi}X^{1/4}} \exp\left(\frac{2}{3}X^{3/2}\right)$ as $X \to \infty$.
- $\mathrm{Bi}(X) \sim \frac{1}{\sqrt{\pi}(-X)^{1/4}} \cos\left(\frac{2}{3}(-X)^{3/2} + \frac{\pi}{4}\right)$ as $X \to -\infty$.

Matching inner ($X \to \infty$) with RH outer ($x \to 1^+$)

- Safer to use an intermediate variable $\hat{x}$ to match:

$$x - 1 = \delta^{\alpha} \hat{x} = \delta X \quad (0 < \alpha < 1)$$

- $X = \frac{\hat{x}}{\delta^{1-\alpha}} \to \infty$ as $\delta \to 0^+$ with $\hat{x} > 0$, $\hat{x} = \mathrm{ord}(1) \Rightarrow$

- $$\delta^{-1/4} Y\left(\frac{\hat{x}}{\delta^{1-\alpha}}\right) \sim \frac{c_3}{\delta^{1/4}} \mathrm{Ai}\left(\frac{\hat{x}}{\delta^{1-\alpha}}\right) + \frac{c_4}{\delta^{1/4}} \mathrm{Bi}\left(\frac{\hat{x}}{\delta^{1-\alpha}}\right)$$
$$\sim \frac{c_3}{\delta^{1/4}} \frac{1}{2\sqrt{\pi}(\hat{x}/\delta^{1-\alpha})^{1/4}} \exp\left(-\frac{2}{3}\left(\frac{\hat{x}}{\delta^{1-\alpha}}\right)^{3/2}\right)$$
$$+ \frac{c_4}{\delta^{1/4}} \frac{1}{\sqrt{\pi}(\hat{x}/\delta^{1-\alpha})^{1/4}} \exp\left(\frac{2}{3}\left(\frac{\hat{x}}{\delta^{1-\alpha}}\right)^{3/2}\right)$$

- $x = 1 + \delta^{\alpha}\hat{x} \to 1^+$ as $\delta \to 0^+$ with $\hat{x} > 0$, $\hat{x} = \mathrm{ord}(1) \Rightarrow$

$$y(1+\delta^{\alpha}\hat{x}) \sim c_1 A_0^R(1+\delta^{\alpha}\hat{x}) \exp\left(-\frac{1}{\varepsilon}\phi^R(1+\delta^{\alpha}\hat{x})\right)$$
$$\sim \frac{c_1}{(2\delta^{\alpha}\hat{x})^{1/4}} \exp\left(-\frac{1}{\varepsilon}\frac{2^{3/2}}{3}(\delta^{\alpha}\hat{x})^{3/2}\right)$$
$$= \frac{c_1}{\delta^{1/4} 2^{1/4} (\hat{x}/\delta^{1-\alpha})^{1/4}} \exp\left(-\frac{2}{3}\left(\frac{\hat{x}}{\delta^{1-\alpha}}\right)^{3/2}\right)$$

- Matching $\Rightarrow c_4 = 0, \quad \frac{c_3}{2\sqrt{\pi}} = \frac{c_1}{2^{1/4}}$

- Hence, $Y_0 = c_3 \mathrm{Ai}(X)$, where $c_3 = \frac{2^{3/4}}{\sqrt{\pi}} c_1$.

Matching inner ($X \to -\infty$) with LH outer ($x \to 1^-$)

- $X = \dfrac{\hat{x}}{\delta^{1-\alpha}} \to -\infty$ as $\delta \to 0^+$ with $\hat{x} < 0$, $\hat{x} = \mathrm{ord}(1) \Rightarrow$

$$\delta^{-1/4} Y\left(\frac{\hat{x}}{\delta^{1-\alpha}}\right) \sim \frac{c_3}{\delta^{1/4}} \mathrm{Ai}\left(\frac{\hat{x}}{\delta^{1-\alpha}}\right)$$

$$\sim \frac{c_3}{\delta^{1/4}} \frac{1}{\sqrt{\pi}\,(-\hat{x}/\delta^{1-\alpha})^{1/4}} \sin\left(\frac{2}{3}\left(-\frac{\hat{x}}{\delta^{1-\alpha}}\right)^{3/2} + \frac{\pi}{4}\right)$$

- $x = 1 + \delta^{\alpha}\hat{x} \to 1^-$ as $\delta \to 0^+$ with $\hat{x} < 0$, $\hat{x} = \mathrm{ord}(1) \Rightarrow$

$$y(1+\delta^{\alpha}\hat{x}) \sim c_2 A_0^R(1+\delta^{\alpha}\hat{x}) \cos\left(\frac{1}{\varepsilon}\phi^L(1+\delta^{\alpha}\hat{x})\right)$$

$$\sim \frac{c_2}{\delta^{1/4} 2^{1/4} (-\hat{x}/\delta^{1-\alpha})^{1/4}} \cos\left(\frac{\pi}{4\varepsilon} - \frac{2}{3}\left(-\frac{\hat{x}}{\delta^{1-\alpha}}\right)^{3/2}\right)$$

- Matching $\Rightarrow$ for $z = \frac{2}{3}\left(-\frac{\hat{x}}{\delta^{1-\alpha}}\right)^{3/2} > 0$,

$$\frac{c_3}{\sqrt{\pi}} \sin\left(z + \frac{\pi}{4}\right) \sim \frac{c_2}{2^{1/4}} \cos\left(\frac{\pi}{4\varepsilon} - z\right) \quad \text{as } \varepsilon \to 0^+$$

$$\Rightarrow \frac{c_3}{\sqrt{\pi}}\left[\sin\frac{\pi}{4}\cos z + \cos\frac{\pi}{4}\sin z\right] \sim \frac{c_2}{2^{1/4}}\left[\cos\frac{\pi}{4\varepsilon}\cos z + \sin\frac{\pi}{4\varepsilon}\sin z\right] \quad \text{as } \varepsilon \to 0^+$$

$$\Rightarrow \quad \frac{c_2}{2^{1/4}}\cos\frac{\pi}{4\varepsilon} \sim \frac{c_3}{\sqrt{\pi}}\sin\frac{\pi}{4}, \quad \frac{c_2}{2^{1/4}}\sin\frac{\pi}{4\varepsilon} \sim \frac{c_3}{\sqrt{\pi}}\cos\frac{\pi}{4} \quad \text{as } \varepsilon \to 0^+$$

$\Rightarrow$ ($c_2, c_3 \neq 0$) $\quad \tan\dfrac{\pi}{4\varepsilon} \sim \cot\dfrac{\pi}{4} = 1$ as $\varepsilon \to 0^+$

$\Rightarrow \quad \dfrac{\pi}{4\varepsilon} \sim \dfrac{\pi}{4} + n\pi$ as $n \to \infty$ with $n \in \mathbb{N}$

$\Rightarrow \quad \lambda_n = \dfrac{1}{\varepsilon_n} \sim 4n + 1$ as $n \to \infty$ with $n \in \mathbb{N}$.

• NB: $\dfrac{c_2(-1)^n}{2^{1/4}\sqrt{2}} = \dfrac{c_3}{\sqrt{2\pi}} \Rightarrow c_1 = \dfrac{c_3}{2^{3/4}\sqrt{\pi}} = \dfrac{(-1)^n c_2}{2}$

and this is called a connection formula.

• NB: agreement with exact solution, namely

$$y_n = e^{-x^2/2} H_{2n}(x), \quad \lambda_n = 4n+1.$$

• Cf: analysis using complex representation of WKB solutions in online notes.