BL AND LOG MATCHING

- Suppose $y(x)$ satisfies $(x^2y')' + \varepsilon x^2 yy' = 0$ for $x>1$, with $y(1)=0$ and $y(\infty)=1$, where $0<\varepsilon \ll 1$.

- Try $y(x) \overset{①}{\sim} y_0(x) + \varepsilon y_2(x) + \ldots$ as $\varepsilon \to 0^+$.

- $O(\varepsilon^0)$: $(x^2 y_0')' = 0$ for $x>1$, with $y_0(1)=0$, $y_0(\infty)=1$

$$\Rightarrow \underline{\underline{y_0(x) = 1 - 1/x}}$$

- $O(\varepsilon^1)$: $(x^2 y_2')' = -x^2 y_0 y_0' \Rightarrow (x^2y_2')' \overset{②}{=} -1 + 1/x$ for $x>1$

$$y_2(1) = 0 \Rightarrow \underline{\underline{y_2(x) \overset{③}{=} A_2(1-1/x) - \ln x - \frac{\ln x}{x} \quad (A_2 \in \mathbb{R})}}$$

- Cannot satisfy $y_2(\infty)=0$ $\because$ $y_2(x) \to -\infty$ as $x\to\infty$ for any $A_2 \in \mathbb{R}$.

- This is a consequence of a term on the RHS of ② being a solution of the homogeneous version of ②, exciting thereby resonant forcing or secular terms in ③ [-1 on RHS ② $\Rightarrow$ $-\ln x$ on RHS ③].

- Hence naive a.e. ① not valid for large x and there is a BL at ∞.

- Since $y(x) \to 1$ as $x\to\infty$, try scaling

$$x = \frac{X}{\delta_1(\varepsilon)}, \quad y(x) = 1 + \delta_2(\varepsilon) Y(X),$$

with $\delta_1(\varepsilon), \delta_2(\varepsilon) \to 0$, $X = \text{ord}(1)$ as $\varepsilon \to 0^+$

- ODE becomes

$$\frac{d}{dX}\left(X^2\frac{dY}{dX}\right)+\frac{\varepsilon}{\delta_1}X\frac{dY}{dX}+\frac{\varepsilon\delta_2}{\delta_1}X^2Y\frac{dY}{dX}=0$$

- Only one possible dominant balance for $\delta_1, \delta_2 \ll 1$, namely between 1st and 2nd terms, so let $\delta_1=\varepsilon$.

- Note δ_2 left undetermined.

- Expand $Y=Y_0(X)+o(1)$ as $\varepsilon\to 0^+$

$$\Rightarrow \qquad \frac{d}{dX}\left(X^2\frac{dY_0}{dX}\right)+X^2\frac{dY_0}{dX}=0$$

$$\Rightarrow \qquad X^2\frac{d^2Y_0}{dX^2}=-(2X+X^2)\frac{dY_0}{dX}$$

$$\Rightarrow \qquad \ln\left|\frac{dY_0}{dX}\right| = \text{const}-2\ln X - X$$

$$\Rightarrow \qquad \frac{dY_0}{dX}=\text{const}\,\frac{e^{-X}}{X^2}$$

- Imposing $Y_0(\infty)=0$ gives

$$\underline{\underline{Y_0(X)=B_0\int_X^\infty \frac{e^{-s}}{s^2}\,ds \qquad (B_0\in\mathbb{R})}}$$

- Integration by parts $\Rightarrow$ for $X>0$,

$$Y_0(X)=B_0\left[\frac{e^{-X}}{X}+e^{-X}\ln X-\underbrace{\int_0^\infty e^{-s}\ln s\,ds}_{=-\gamma\approx 0.57,\text{ Euler's constant}}+\int_0^X e^{-s}\ln s\,ds\right]$$

$$\Rightarrow Y_0(X)=B_0\left[\frac{1}{X}+\ln X+\gamma-1-\frac{1}{2}X\right]+O(X^2)\text{ as }X\to 0^+$$

Matching via an intermediate variable

- Let $x = \frac{\hat{x}}{\varepsilon^{\alpha}} = \frac{X}{\varepsilon}$ for $0 < \alpha < 1$.

- As $\varepsilon \to 0^+$ with $\hat{x} > 0$, $\hat{x} = \text{ord}(1)$:

$$y(x = \varepsilon^{-\alpha}\hat{x}) \sim y_0(\varepsilon^{-\alpha}\hat{x}) = 1 - \frac{\varepsilon^{\alpha}}{\hat{x}},$$

while

$$1 + \delta_2(\varepsilon) Y(X = \varepsilon^{1-\alpha}\hat{x}) \sim 1 + B_0 \delta_2(\varepsilon)\left[\frac{\varepsilon^{\alpha-1}}{\hat{x}}\right] + \cdots.$$

- Matching $\Rightarrow$ $\delta_2(\varepsilon) = \varepsilon$ (wlog),

$$\underline{\underline{B_0 = -1.}}$$

- Hence,

$$1 + \delta_2(\varepsilon) Y(\varepsilon^{1-\alpha}\hat{x}) \overset{④}{\sim} 1 - \varepsilon\left[\frac{\varepsilon^{\alpha-1}}{\hat{x}} + \ln(\varepsilon^{-\alpha+1}\hat{x}) + \gamma - 1\right] + \cdots$$

as $\varepsilon \to 0^+$ with $\hat{x} > 0$, $\hat{x} = \text{ord}(1)$, so we should have expanded

$$y(x) \sim y_0(x) + \varepsilon \ln\tfrac{1}{\varepsilon}\, y_1(x) + \varepsilon y_2(x) + \cdots$$

as $\varepsilon \to 0^+$ with $x > 1$, $x = \text{ord}(1)$ (instead of ①).

- This gives $y_0(x)$ and $y_2(x)$ as before, and at

$O(\varepsilon \ln\frac{1}{\varepsilon})$: $(x^2 y_1')' = 0$ for $x > 1$, with $y_1(1) = 0$

$$\Rightarrow y_1(x) = A_1(1 - 1/x) \quad (A_1 \in \mathbb{R})$$

• Hence, as $\varepsilon \to 0^+$ with $\hat{x} > 0$, $\hat{x} = \mathrm{ord}(1)$

$$y(x = \varepsilon^{-\alpha}\hat{x}) \sim y_0(\varepsilon^{-\alpha}\hat{x}) + \varepsilon \ln\tfrac{1}{\varepsilon}\, y_1(\varepsilon^{-\alpha}\hat{x}) + \varepsilon y_2(\varepsilon^{-\alpha}\hat{x}) + \dots$$

$$\overset{⑤}{\sim} 1 - \varepsilon\left[\frac{\varepsilon^{\alpha-1}}{\hat{x}} + (1-A_1)\ln\tfrac{1}{\varepsilon} + \ln(\varepsilon^{-\alpha+1}\hat{x}) - A_2\right] + \dots$$

• Matching ④, ⑤ $\Rightarrow$ $\underline{\underline{A_1 = 1,\ A_2 = 1-\gamma}}$

• Hence, for $x > 1$, $x = \mathrm{ord}(1)$,

$$\underline{\underline{y \sim \left(1-\tfrac{1}{x}\right) + \varepsilon\ln\tfrac{1}{\varepsilon}\left(1-\tfrac{1}{x}\right) + \varepsilon\left(-\ln x - \frac{\ln x}{x} + (1-\gamma)\left(1-\tfrac{1}{x}\right)\right) + \dots,}}$$

while for $X = \varepsilon x > 0$, $X = \varepsilon x = \mathrm{ord}(1)$,

$$\underline{\underline{y \sim 1 - \varepsilon\int_X^\infty \frac{e^{-s}}{s^2}\,ds + \dots}}\ , \text{ both as } \varepsilon \to 0^+.$$

• NB: Van Dyke's matching rule works if we treat $\ln\frac{1}{\varepsilon}$ as being of $\mathrm{ord}(1)$ for the purposes of matching.

• (2.t.i.) in outer variables $= y_0(X/\varepsilon) + \varepsilon\ln\frac{1}{\varepsilon}\, y_1(X/\varepsilon) + \varepsilon y_2(X/\varepsilon)$

$$\Rightarrow (2.t.o.)(2.t.i.) \quad = 1 + \varepsilon\ln\tfrac{1}{\varepsilon}(A_1 - 1) + \varepsilon\left(-\tfrac{1}{X} - \ln X + A_2\right)$$

• (2.t.o.) in inner variables $= 1 + \varepsilon Y_0(\varepsilon x)$

$$\Rightarrow (2.t.i.)(2.t.o.) \quad = 1 + \varepsilon B_0\left(\tfrac{1}{\varepsilon x} + \ln \varepsilon x + \gamma - 1\right)$$

$$= 1 + \varepsilon B_0\left(\tfrac{1}{X} + \ln X + \gamma - 1\right)$$

• (2.t.o.)(2.t.i.) = (2.t.i.)(2.t.o.) $\Rightarrow$ $\underline{\underline{A_1 = 1,\ B_0 = -1,\ A_2 = 1-\gamma, \text{ as before}}}$