MATCHED ASYMPTOTIC EXPANSIONS AND BOUNDARY LAYER THEORY.

HIGHER ORDER MATCHING:

- Substitute leading-order solution into second-order solution $\Rightarrow$

$$y_{L1}(x_L) = -ebx_L + (a-eb)x_L e^{-x_L} + A_{L1} + B_{L1}e^{-x_L},$$

$$y_{M1}(x) = -ebxe^{-x} + A_{M1}e^{-x},$$

$$y_{R1}(x_R) = -bx_R + A_{R1} + B_{R1}e^{-x_R},$$

where $A_{L1}, B_{L1}, A_{M1}, A_{R1}, B_{R1} \in \mathbb{R}$.

- Recall BCs $\Rightarrow$ $\quad y_{L1}(0) = 0 \Rightarrow A_{L1} + B_{L1} = 0$

$$y_{R1}(0) = 0 \Rightarrow A_{R1} + B_{R1} = 0$$

Matching LHBL and middle region

- As $\varepsilon \to 0^+$ with $\hat{x} > 0$ and $\hat{x} = \text{ord}(1)$,

$$y_L(x_L = \varepsilon^{\alpha-1}\hat{x}) = y_{L0}(\varepsilon^{\alpha-1}\hat{x}) + \varepsilon y_{L1}(\varepsilon^{\alpha-1}\hat{x}) + O(\varepsilon^2)$$

$$= eb + (a-eb)e^{-\varepsilon^{\alpha-1}\hat{x}} + \varepsilon\left[-eb\varepsilon^{\alpha-1}\hat{x} + (a-eb)\varepsilon^{\alpha-1}\hat{x}e^{-\varepsilon^{\alpha-1}\hat{x}} + A_{L1} + B_{L1}e^{-\varepsilon^{\alpha-1}\hat{x}}\right] + O(\varepsilon^2)$$

$$= eb - \varepsilon^{\alpha}eb\hat{x} + \varepsilon A_{L1} + O(\varepsilon^2),$$

absorbing the E.S.T.'s into the $O(\varepsilon^2)$ term.

- As $\varepsilon \to 0^+$ with $\hat{x} > 0$ and $\hat{x} = \mathrm{ord}(1)$,

$$y_M(x = \varepsilon^\alpha \hat{x}) = y_{M0}(\varepsilon^\alpha \hat{x}) + \varepsilon y_{M1}(\varepsilon^\alpha \hat{x}) + O(\varepsilon^2)$$

$$= eb e^{-\varepsilon^\alpha \hat{x}} + \varepsilon\left[-eb\varepsilon^\alpha \hat{x} e^{-\varepsilon^\alpha \hat{x}} + A_{M1} e^{-\varepsilon^\alpha \hat{x}}\right] + O(\varepsilon^2)$$

$$= eb\left[1 - \varepsilon^\alpha \hat{x} + O(\varepsilon^{2\alpha})\right] + \varepsilon\left[-eb\varepsilon^\alpha \hat{x}(1 - \varepsilon^\alpha \hat{x} + O(\varepsilon^{2\alpha})) + A_{M1}(1 - \varepsilon^\alpha \hat{x} + O(\varepsilon^{2\alpha}))\right] + O(\varepsilon^2)$$

$$= eb - \varepsilon^\alpha eb\hat{x} + \varepsilon A_{M1} + O(\varepsilon^{2\alpha}, \varepsilon^{\alpha+1}, \varepsilon^2) \qquad (1/2 < \alpha < 1)$$

- Same expansions $\Rightarrow A_{L1} = A_{M1}$

- Note a term "jumps order": $-\varepsilon^\alpha eb\hat{x}$ comes from $y_{L0}(\varepsilon^{\alpha-1}\hat{x})$ & $y_{M1}(\varepsilon^\alpha \hat{x})$.

Matching RHBL and middle region

- As $\varepsilon \to 0^+$ with $\hat{x} < 0$ and $\hat{x} = \mathrm{ord}(1)$,

$$y_R(x_R = \varepsilon^{\alpha-1}\hat{x}) = y_{R0}(\varepsilon^{\alpha-1}\hat{x}) + \varepsilon y_{R1}(\varepsilon^{\alpha-1}\hat{x}) + O(\varepsilon^2)$$

$$= b + \varepsilon\left[-b\varepsilon^{\alpha-1}\hat{x} + A_{R1} + B_{R1} e^{-\varepsilon^{\alpha-1}\hat{x}}\right] + O(\varepsilon^2)$$

$$= \varepsilon B_{R1} e^{-\varepsilon^{\alpha-1}\hat{x}} + b - \varepsilon^\alpha b\hat{x} + \varepsilon A_{R1} + O(\varepsilon^2)$$

- Note $\varepsilon B_{R1} e^{-\varepsilon^{\alpha-1}\hat{x}}$ an E.L.T. for $B_{R1} = \mathrm{ord}(\varepsilon^m)\ \forall m > 0$.

- As $\varepsilon \to 0^+$ with $\hat{x} < 0$ and $\hat{x} = \mathrm{ord}(1)$,

$$y_M(x = 1 + \varepsilon^\alpha \hat{x}) = y_{M0}(1 + \varepsilon^\alpha \hat{x}) + \varepsilon y_{M1}(1 + \varepsilon^\alpha \hat{x}) + O(\varepsilon^2)$$

$$= ebe^{-(1+\varepsilon^\alpha \hat{x})} + \varepsilon\left[-eb(1+\varepsilon^\alpha \hat{x})e^{-(1+\varepsilon^\alpha \hat{x})} + A_{M1}e^{-(1+\varepsilon^\alpha \hat{x})}\right] + O(\varepsilon^2)$$

$$= b\left[1 - \varepsilon^\alpha \hat{x} + O(\varepsilon^{2\alpha})\right] + \varepsilon\left[-b(1+\varepsilon^\alpha \hat{x})(1 - \varepsilon^\alpha \hat{x} + O(\varepsilon^{2\alpha})) + A_{M1}e^{-1}(1 - \varepsilon^\alpha \hat{x} + O(\varepsilon^{2\alpha}))\right] + O(\varepsilon^2)$$

$$= b - \varepsilon^\alpha b\hat{x} - \varepsilon b + \varepsilon A_{M1}/e + O(\varepsilon^{2\alpha}, \varepsilon^{\alpha+1}, \varepsilon^2)$$

$(1/2 < \alpha < 1)$

- Same expansions $\Rightarrow$ $B_{R1} = 0$, $A_{R1} = A_{M1}/e - b$.

Second-order solution

- BCs $\Rightarrow$ $A_{L1} + B_{L1} = 0$, $A_{R1} + B_{R1} = 0$
- Matching $\Rightarrow$ $A_{L1} = A_{M1}$, $B_{R1} = 0$, $A_{R1} = A_{M1}/e - b$
- Hence, $A_{R1} = 0$, $B_{R1} = 0$, $A_{M1} = eb$, $A_{L1} = eb$, $B_{L1} = -eb$, giving

$$y_{L1}(x_L) = -ebx_L + (a - eb)x_L e^{-x_L} + eb - ebe^{-x_L},$$

$$y_{M1}(x) = -ebxe^{-x} + ebe^{-x},$$

$$y_{R1}(x_R) = -bx_R.$$

- Note variation in y in RHBL follows from Taylor expansion of $y_M(1 + \varepsilon x_R)$ $\Rightarrow$ RHBL not needed up to $O(\varepsilon)$.

Van Dyke's matching 'rule'

- Using the intermediate variable $\hat{x}$ can be tiresome.
- Van Dyke's matching rule is quicker and usually works:

$$(m.\ t.\ i.)(n.\ t.\ o.) = (n.\ t.\ o.)(m.\ t.\ i.)$$

i.e. n terms in outer expansion, written in inner variables, and reexpanded to m terms, is the same as m terms in the inner expansion, written in outer variables, and reexpanded to n terms.

Example again: leading-order matching

- $(1.\ t.\ o.) = A_{M0} e^{-x}$

$\Rightarrow$ $(1.\ t.\ o.)$ in inner variables $= A_{M0} e^{-\varepsilon x_L} = A_{M0} - \varepsilon A_{M0} x_L + O(\varepsilon^2)$
as $\varepsilon \to 0^+$ with $x_L = \mathrm{ord}(1)$

$\Rightarrow (1.\ t.\ i.)(1.\ t.\ o.) = A_{M0}$

- $(1.\ t.\ i.) = A_{L0} + B_{L0} e^{-x_L}$

$\Rightarrow$ $(1.\ t.\ i.)$ in outer variables $= A_{L0} + B_{L0} e^{-x/\varepsilon} = A_{L0} + E.S.T$
as $\varepsilon \to 0^+$ with $x > 0$, $x = \mathrm{ord}(1)$

$\Rightarrow (1.\ t.\ o.)(1.\ t.\ i.) = A_{L0}$

- Thus $(1.t.i.)(1.t.o.) = (1.t.o.)(1.t.i.) \Rightarrow A_{M0} = A_{L0}$.

Example again: second-order matching

- $(2.t.o.) \qquad = A_{M0}e^{-x} + \varepsilon\left(A_{M1}e^{-x} - A_{M0}xe^{-x}\right)$

$$\Rightarrow (2.t.o.) \text{ in inner variables } = A_{M0}e^{-\varepsilon x_L} + \varepsilon\left(A_{M1}e^{-\varepsilon x_L} - A_{M0}\varepsilon x_L e^{-\varepsilon x_L}\right)$$
$$= A_{M0} - \varepsilon A_{M0}x_L + \varepsilon A_{M1} + O(\varepsilon^2)$$
$$\text{as } \varepsilon \to 0^+ \text{ with } x_L = \text{ord}(1)$$

$$\Rightarrow (2.t.i.)(2.t.o.) \qquad = A_{M0} + \varepsilon(-A_{M0}x_L + A_{M1})$$

- $(2.t.i.) \qquad = A_{L0} + B_{L0}e^{-x_L} + \varepsilon\left(A_{L1} + B_{L1}e^{-x_L} - A_{L0}x_L + B_{L0}x_Le^{-x_L}\right)$

$$\Rightarrow (2.t.i.) \text{ in outer variables } = A_{L0} + B_{L0}e^{-x/\varepsilon} + \varepsilon\left(A_{L1} + B_{L1}e^{-x/\varepsilon} - A_{L0}\frac{x}{\varepsilon} + B_{L0}\frac{x}{\varepsilon}e^{-x/\varepsilon}\right)$$
$$= A_{L0} - A_{L0}x + \varepsilon A_{L1} + \text{E.S.T.}$$
$$\text{as } \varepsilon \to 0^+ \text{ with } x > 0,\ x = \text{ord}(1)$$

$$\Rightarrow (2.t.o.)(2.t.i.) \qquad = A_{L0} - A_{L0}x + \varepsilon A_{L1}$$

$$\Rightarrow (2.t.o.)(2.t.i.) \text{ in inner variables } = A_{L0} + \varepsilon(-A_{L0}x_L + A_{L1})$$

- Thus, $(2.t.i.)(2.t.o.) = (2.t.o.)(2.t.i.) \Rightarrow A_{M0} = A_{L0},\ A_{M1} = A_{L1}$

- <u>Ex</u>: repeat for $(m, n) = (1, 2)$ and $(2, 1)$.

- NB: if possible, try to match only at break where the power of ε changes, i.e. treat $\log\varepsilon$ as being of ord(1) for the purposes of matching.

Composite expansion

- Aim to combine inner and outer expansions to obtain a uniformly valid expansion (for plotting etc).

- One way is the additive composite expansion ($p \in \mathbb{N}$):

$$y_c = (\text{p.t.o.}) + (\text{p.t.i.}) - (\text{p.t.i.})(\text{p.t.o.})$$

when it exists, which gives a uniformly valid expansion, with error $O(\varepsilon^p)$ if inner and outer expansions are in powers of ε.

- Subtract $(\text{p.t.i.})(\text{p.t.o.}) = (\text{p.t.o.})(\text{p.t.i.})$ because it has been counted twice in overlap region.

Example again

- $p = 1 \Rightarrow$
$$\begin{aligned} y_c &= y_{M0}(x) + y_{L0}(x/\varepsilon) - (\text{1.t.i.})(\text{1.t.o.}) \\ &= ebe^{-x} + eb + (a - eb)e^{-x/\varepsilon} - eb \\ &= ebe^{-x} + (a - eb)e^{-x/\varepsilon}. \end{aligned}$$

- $p = 2 \Rightarrow$
$$\begin{aligned} y_c &= y_{M0}(x) + \varepsilon y_{M1}(x) + y_{L0}(x/\varepsilon) + \varepsilon y_{L1}(x/\varepsilon) - (\text{2.t.i.})(\text{2.t.o.}) \\ &= (a - \varepsilon b)e^{-x/\varepsilon} + (a - eb)xe^{-x/\varepsilon} - \varepsilon ebe^{-x/\varepsilon} \\ &\quad + ebe^{-x} - \varepsilon ebxe^{-x} + \varepsilon ebe^{-x} \end{aligned}$$

because $(\text{2.t.i.})(\text{2.t.o.}) = A_{M0} - A_{M0}x + \varepsilon A_{M1} = eb - ebx + \varepsilon eb$

- NB: composite expansion not unique $\because$ not of Poincaré form.

Choice of rescaling revisited

- In LHBL began with scaling $x = \varepsilon^{\alpha} x_L$, $y(x) = y_L(x_L) \Rightarrow$

$$\varepsilon^{1-2\alpha} \frac{d^2 y_L}{dx_L^2} + \varepsilon^{-\alpha} \frac{dy_L}{dx_L} + y_L = 0$$

$\alpha = 0$	balance	outer
$0 < \alpha < 1$	dominant	overlap
$\alpha = 1$	balance	inner
$\alpha > 1$	dominant	sub-inner

- The inner and outer regions can be matched because they share a common term, which is dominant in the overlap region.

- Two dominant balances: $\alpha = 0$ (outer) and $\alpha = 1$ (inner), corresponding to the distinguished limits in which $x = \text{ord}(1)$ and $x = \text{ord}(\varepsilon)$, respectively.