MATCHED ASYMPTOTIC EXPANSIONS AND BOUNDARY LAYER THEORY.

SINGULAR PERTURBATION:

- e.g. $\varepsilon y'' + y' + y = 0$ for $0 < x < 1$, with $y(0) = a$, $y(1) = b$.

  When $\varepsilon = 0$, $y' + y = 0 \Rightarrow y = Ae^{-x}$ $(A \in \mathbb{R})$, which cannot satisfy both BCs in general.

- This is a singular perturbation problem

- Suppose D_ε is a differential operator that depends on a small parameter, e.g.

$$D_\varepsilon = \varepsilon \frac{d^2}{dx^2} + \frac{d}{dx} + 1$$

- Then the problem "$D_\varepsilon y = 0$ + BCs" is a singular perturbation problem if the order of $D_0 y = 0$ is less than that of $D_\varepsilon y = 0$ as $\varepsilon \to 0$, because the solution of $D_0 y = 0$ cannot in general satisfy all of the BCs.

- Suppose $D_\varepsilon = \varepsilon \frac{d^k}{dx^k}$ + lower order derivatives.

- Then usually,

  - over most of range $\varepsilon \frac{d^k y}{dx^k}$ is small and y satisfies $D_0 y = 0$ at leading order;

  - in certain regions (often near boundaries), $\varepsilon \frac{d^k y}{dx^k}$ is not small and y adjusts itself to satisfy BCs.

- Regions in which $\frac{d^k y}{dx^k}$ is large are known as boundary, edge and skin layers in fluid dynamics, solid mechanics and electrodynamics.

- The usual procedure for finding a solution to a singular ODE problem is as follows:

  (1) determine the scaling of the boundary layers (e.g. $x = \mathrm{ord}(\varepsilon)$, $x = \mathrm{ord}(\varepsilon^{1/2})$ etc);

  (2) rescale the dependent variable in the boundary layer (e.g. $x = \varepsilon\hat{x}$, $\hat{x} = \mathrm{ord}(1)$ etc);

  (3) find the asymptotic expansions of the solutions in the boundary layers and outside them (i.e. the "inner" and "outer" solutions);

  (4) fix the constants of integration in these solutions by

    (a) demanding the inner solutions obey the BCs;

    (b) making the asymptotic expansions of the outer and inner solutions agree in an overlap or transition region between them - i.e. "matching".

- This is called the method of matched asymptotic expansions.

- Main e.g. $\varepsilon y'' + y' + y = 0$ for $0 < x < 1$, with $y(0) = a$, $y(1) = b$.

LH boundary layer scaling

- $x = \varepsilon^{\alpha} x_L$, $y(x) = y_L(x_L)$, where $\alpha > 0$

$$\Rightarrow \quad y' = \frac{dy_L}{dx_L}\frac{dx_L}{dx} = \varepsilon^{-\alpha}\frac{dy_L}{dx_L}$$

$$\Rightarrow \quad \varepsilon^{1-2\alpha}\frac{d^2 y_L}{dx_L^2} + \varepsilon^{-\alpha}\frac{dy_L}{dx_L} + y_L = 0$$

- To satisfy BC at $x = 0$ and match with outer solution, need the $\frac{d^2 y_L}{dx_L^2}$ term to be in the <u>dominant balance</u>

$\Rightarrow$ need to balance this term with the largest of the others

$\Rightarrow$ $\alpha = 1$, i.e. the BL has width of $\text{ord}(\varepsilon)$.

- Similar argument $\Rightarrow$ RH BL near $x = 1$ has width of $\text{ord}(\varepsilon)$.

Develop asymptotic solution as follows:

(1) away from ends of interval (the "middle") expand

$$y(x) = y_M(x) \sim y_{M0}(x) + \varepsilon y_{M1}(x) + \cdots \quad \text{as } \varepsilon \to 0^+, \text{ with } x, 1-x = \text{ord}(1);$$

(2) near LH end scale $x = \varepsilon x_L$ and expand

$$y(x) = y_L(x_L) \sim y_{L0}(x_L) + \varepsilon y_{L1}(x_L) + \cdots \quad \text{as } \varepsilon \to 0^+, \text{ with } 0 < x_L = \text{ord}(1);$$

(3) near RH end scale $x = 1 + \varepsilon x_R$ and expand

$$y(x) = y_R(x_R) \sim y_{R0}(x_R) + \varepsilon y_{R1}(x_R) + \cdots \quad \text{as } \varepsilon \to 0^+, \text{ with } 0 < -x_R = \text{ord}(1).$$

LH BL

ODE $\Rightarrow$ $$\frac{d^2y_L}{dx_L^2} + \frac{dy_L}{dx_L} + \varepsilon y_L = 0 \quad \text{for } x_L > 0.$$

Expansion $\Rightarrow$ $$O(\varepsilon^0): \quad \frac{d^2y_{L0}}{dx_L^2} + \frac{dy_{L0}}{dx_L} = 0 \quad \text{for } x_L > 0$$

$$O(\varepsilon^1): \quad \frac{d^2y_{L1}}{dx_L^2} + \frac{dy_{L1}}{dx_L} = -y_{L0} \quad \text{for } x_L > 0 \text{ etc}$$

Solve $\Rightarrow$ $$y_{L0}(x_L) = A_{L0} + B_{L0}e^{-x_L} \qquad (A_{L0}, B_{L0} \in \mathbb{R})$$

$$y_{L1}(x_L) = A_{L1} + B_{L1}e^{-x_L} - A_{L0}x_L + B_{L0}x_Le^{-x_L}$$
$$(A_{L1}, B_{L1} \in \mathbb{R}) \text{ etc}$$

BC $y(0) = a$ $\Rightarrow$ $$y_{L0}(0) = a, \quad y_{L1}(0) = 0 \quad \text{etc}$$

$$\Rightarrow \quad A_{L0} + B_{L0} = a, \quad A_{L1} + B_{L1} = 0 \quad \text{etc}$$

Middle

ODE $\Rightarrow$ $$\varepsilon\frac{d^2y_M}{dx^2} + \frac{dy_M}{dx} + y_M = 0 \quad \text{for } 0 < x < 1$$

Expansion $\Rightarrow$ $$O(\varepsilon^0): \quad \frac{dy_{M0}}{dx} + y_{M0} = 0 \quad \text{for } 0 < x < 1$$

$$O(\varepsilon^1): \quad \frac{dy_{M1}}{dx} + y_{M1} = -\frac{d^2y_{M0}}{dx^2} \quad \text{for } 0 < x < 1 \text{ etc}$$

Solve $\Rightarrow$ $$y_{M0}(x) = A_{M0}e^{-x} \qquad (A_{M0} \in \mathbb{R})$$

$$y_{M1}(x) = A_{M1}e^{-x} - A_{M0}xe^{-x} \quad (A_{M1} \in \mathbb{R}) \text{ etc}$$

Instead of applying BCs at $x = 0, 1$, need to match with expansion in LH BL and in RH BL.

RH BL

ODE $\Rightarrow$ $$\frac{d^2y_R}{dx_R^2} + \frac{dy_R}{dx_R} + \varepsilon y_R = 0 \quad \text{for } x_R < 0$$

Expansion $\Rightarrow$ $$O(\varepsilon^0): \quad \frac{d^2y_{R0}}{dx_R^2} + \frac{dy_{R0}}{dx_R} = 0 \quad \text{for } x_R < 0$$

$$O(\varepsilon^1): \quad \frac{d^2y_{R1}}{dx_R^2} + \frac{dy_{R1}}{dx_R} = -y_{R0} \quad \text{for } x_R < 0 \quad \text{etc}$$

Solve $\Rightarrow$ $$y_{R0}(x_R) = A_{R0} + B_{R0}e^{-x_R} \quad (A_{R0}, B_{R0} \in \mathbb{R})$$

$$y_{R1}(x_R) = A_{R1} + B_{R1}e^{-x_R} - A_{R0}x_R + B_{R0}x_R e^{-x_R}$$
$$(A_{R1}, B_{R1} \in \mathbb{R}) \text{ etc}$$

BC $y(1) = 6$ $\Rightarrow$ $$y_{R0}(0) = 6, \quad y_{R1}(0) = 0$$

$$\Rightarrow \quad A_{R0} + B_{R0} = 6, \quad A_{R1} + B_{R1} = 0 \quad \text{etc}$$

Leading-order matching

- The leading-order solutions $y_{L0}(x_L)$, $y_{M0}(x)$ and $y_{R0}(x_R)$ contain 5 arbitrary constants ($A_{L0}, B_{L0}, A_{M0}, A_{R0}, B_{R0}$) and BCs give 2 equations relating them: the other 3 equations come from matching.

- Idea is that between adjacent regions there is an "overlap" or "intermediate" region in which both expansions should hold.

- To match y_L and y_M introduce an intermediate scaling $x = \varepsilon^{\alpha}\hat{x}$, with $0 < \alpha < 1$, then as $\varepsilon \to 0^+$ with $\hat{x} > 0$ and $\hat{x} = \text{ord}(1)$, $x = \varepsilon^{\alpha}\hat{x} \to 0^+$ and $x_L = \varepsilon^{\alpha-1}\hat{x} \to +\infty$.

- Matching requires expansions to be equal as $\varepsilon \to 0^+$ with $\hat{x} > 0$ and $\hat{x} = \text{ord}(1)$, i.e. $y_L(\varepsilon^{\alpha-1}\hat{x}) \sim y_M(\varepsilon^{\alpha}\hat{x})$ as $\varepsilon \to 0^+$ with $\hat{x} > 0$ and $\hat{x} = \text{ord}(1)$.

- As $\varepsilon \to 0^+$ with $\hat{x} > 0$ and $\hat{x} = \text{ord}(1)$,

$$y_L(\varepsilon^{\alpha-1}\hat{x}) = y_{L0}(\varepsilon^{\alpha-1}\hat{x}) + O(\varepsilon)$$

$$= A_{L0} + \underbrace{B_{L0}e^{-\varepsilon^{\alpha-1}\hat{x}}}_{E.S.T.} + O(\varepsilon)$$

and

$$y_M(\varepsilon^{\alpha}\hat{x}) = y_{M0}(\varepsilon^{\alpha}\hat{x}) + O(\varepsilon)$$

$$= A_{M0}e^{-\varepsilon^{\alpha}\hat{x}} + O(\varepsilon)$$

$$= A_{M0}(1 - \varepsilon^{\alpha}\hat{x} + O(\varepsilon^{2\alpha})) + O(\varepsilon)$$

- Same expansions $\Rightarrow$ $A_{L0} = A_{M0}$, i.e. $y_{L0}(+\infty) = y_{M0}(0^+)$ and the outer limit of the inner solution matches with the inner limit of the outer solution.

- On RHS let $x = 1 + \varepsilon^{\alpha}\hat{x}$, where $0 < \alpha < 1$.

- As $\varepsilon \to 0^+$ with $\hat{x} < 0$ and $\hat{x} = \text{ord}(1)$,

$$y_R(x_R = \varepsilon^{\alpha-1}\hat{x}) = y_{R0}(\varepsilon^{\alpha-1}\hat{x}) + O(\varepsilon)$$

$$= A_{R0} + \underbrace{B_{R0}e^{-\varepsilon^{\alpha-1}\hat{x}}}_{E.L.T.} + O(\varepsilon)$$

and

$$y_M(x = 1 + \varepsilon^{\alpha}\hat{x}) = y_{M0}(1 + \varepsilon^{\alpha}\hat{x}) + O(\varepsilon)$$

$$= A_{M0}e^{-(1+\varepsilon^{\alpha}\hat{x})} + O(\varepsilon)$$

$$= \frac{A_{M0}}{e}(1 - \varepsilon^{\alpha}\hat{x} + O(\varepsilon^{2\alpha})) + O(\varepsilon)$$

- Same expansions $\Rightarrow$ $B_{R0} = 0$ and $A_{R0} = \dfrac{A_{M0}}{e}$

Leading-order solution

- BCs $\Rightarrow$ $A_{L0} + B_{L0} = a$, $A_{R0} + B_{R0} = b$
- Matching $\Rightarrow$ $A_{L0} = A_{M0}$, $A_{R0} = \frac{A_{M0}}{e}$, $B_{R0} = 0$
- Hence, $A_{L0} = eb$, $B_{L0} = a - eb$, $A_{M0} = eb$, $A_{R0} = b$, giving

$$y_{L0}(x_L) = eb + (a - eb)e^{-x_L},$$

$$y_{M0}(x) = ebe^{-x},$$

$$y_{R0}(x_R) = b.$$

- No rapid variation in y in RHBL $\Rightarrow$ not needed, at least at leading order.

Agreement with exact solution

- Exact solution is

$$y(x) = A_+ e^{\lambda_- x} - A_- e^{\lambda_+ x} \quad \text{for } 0 \leq x \leq 1,$$

where $A_\pm = \dfrac{ae^{\lambda_\pm} - b}{e^{\lambda_+} - e^{\lambda_-}}$, $\lambda_\pm = \dfrac{-1 \pm \sqrt{1 - 4\varepsilon}}{2\varepsilon}$.

- Using expansions $\lambda_- = -\frac{1}{\varepsilon} + 1 + O(\varepsilon)$ and $\lambda_+ = -1 + O(\varepsilon)$ as $\varepsilon \to 0^+$, can show

$$y(\varepsilon x_L) = y_{L0}(x_L) + O(\varepsilon) \quad \text{as } \varepsilon \to 0^+ \text{ with } x_L > 0,\ x_L = \text{ord}(1);$$

$$y(x) = y_{M0}(x) + O(\varepsilon) \quad \text{as } \varepsilon \to 0^+ \text{ with } 0 < x < 1;\ x, 1 - x = \text{ord}(1)$$

$$y(1 + \varepsilon x_R) = y_{R0}(x_R) + O(\varepsilon) \quad \text{as } \varepsilon \to 0^+ \text{ with } x_R < 0,\ x_R = \text{ord}(1).$$