PERTURBATION METHODS IN ENGINEERING
LECTURE 5

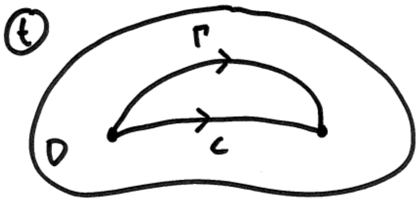
METHODS OF DEEPEST DESCENTS

- Generalizes Laplace's method to deal with

$$I(x) = \int_C f(t)\, e^{x\phi(t)}\, dt$$

as $x \to \infty$, where $f(t), \phi(t)$ are complex and C is a contour in the complex t-plane.

- Key idea: $I(x)$ unchanged upon deforming C to a new contour Γ provided f and ϕ are holomorphic in a domain D containing C, Γ and their interior:



$$I(x) = \int_\Gamma f(t)\, e^{x\phi(t)}\, dt$$

- If we can find a Γ on which $\mathrm{Im}(\phi)$ is piecewise constant, i.e. Γ_j, v_j s.t. $\Gamma = \bigcup_j \Gamma_j$ and $\mathrm{Im}(\phi) = v_j = \text{const. on } \Gamma_j$, then

$$I(x) = \sum_j e^{ixv_j} \int_{\Gamma_j} f(t)\, e^{x\,\mathrm{Re}(\phi(t))}\, dt,$$

each of which we can analyse as $x \to \infty$ using Laplace's method.

- Let $\phi(t) = u(\xi,\eta) + i v(\xi,\eta)$, $t = \xi + i\eta$.

- ϕ holomorphic $\Rightarrow$ Cauchy-Riemann equations (CREs)

$$u_\xi = v_\eta, \quad u_\eta = -v_\xi$$

- CREs $\Rightarrow \nabla u \cdot \nabla v = (u_\xi, u_\eta)\cdot(v_\xi, v_\eta) = 0.$

- Thus, $\nabla v \perp v = \text{const} \Rightarrow \nabla u \ /\!/$ to $v = \text{const}.$

- $\underline{s}$ a unit vector making an angle θ with ∇u

$\Rightarrow \dfrac{\partial u}{\partial s} = (\underline{s}\cdot\nabla)u = |\underline{s}|\,|\nabla u|\cos\theta$

$(-\pi < \theta \leq \pi$ wlog$)$

$\Rightarrow \nabla u$ points in direction of steepest ascent on surface of u $(\theta = 0)$
$-\nabla u$ points in direction of steepest descent on surface of u $(\theta = \pi)$

$\Rightarrow v = \text{const.}$ is a path of steepest ascent/descent on surface of u

- What can we say about landscape of surface $u(\xi, \eta)$?

- CREs $\Rightarrow u_{\xi\xi} + u_{\eta\eta} = (v_\eta)_\xi + (-v_\xi)_\eta = 0.$

- Hence, u can have no maxima or minima (except at singular or branch points where ϕ is not holomorphic), only saddle points (at which $\nabla u = 0$, $\nabla v = 0$, $\dfrac{d\phi}{dt} = 0$).

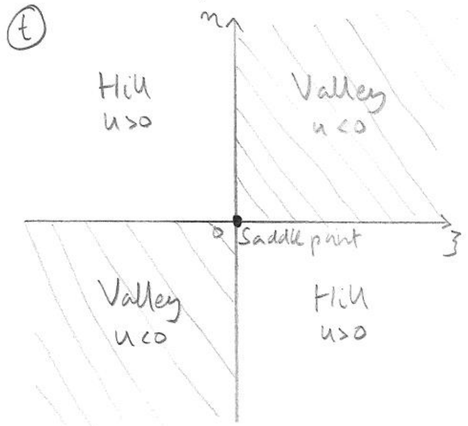
- Hence, landscape of u consists of hills $(u > 0)$ and valleys $(u < 0)$ at infinity, with saddle points in the passes between them.

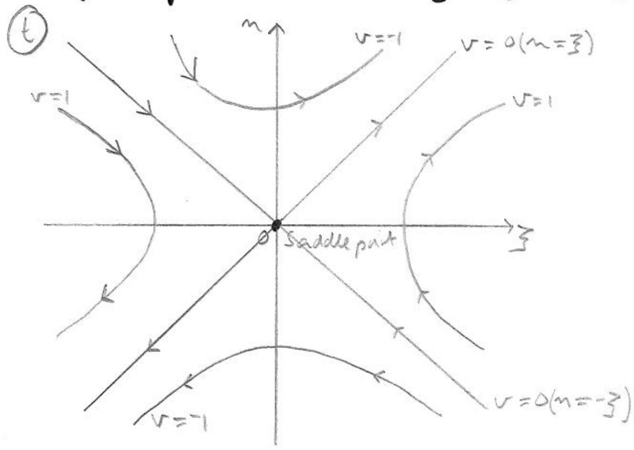
- Note that steepest paths can only intersect at a saddle, singular or branch point because elsewhere $\exists!$ trajectory because $\nabla u \neq \underline{0}, \infty.$

Example

- $\phi = it^2 = i(\xi + i\eta)^2 \Rightarrow \phi' = 2it,\ u = -2\xi\eta,\ v = \xi^2 - \eta^2,\ -\nabla u = 2(\eta, \xi)$

- Thus, saddle point at $t = 0$ and steepest paths are $v = \xi^2 - \eta^2 = \text{const}$.





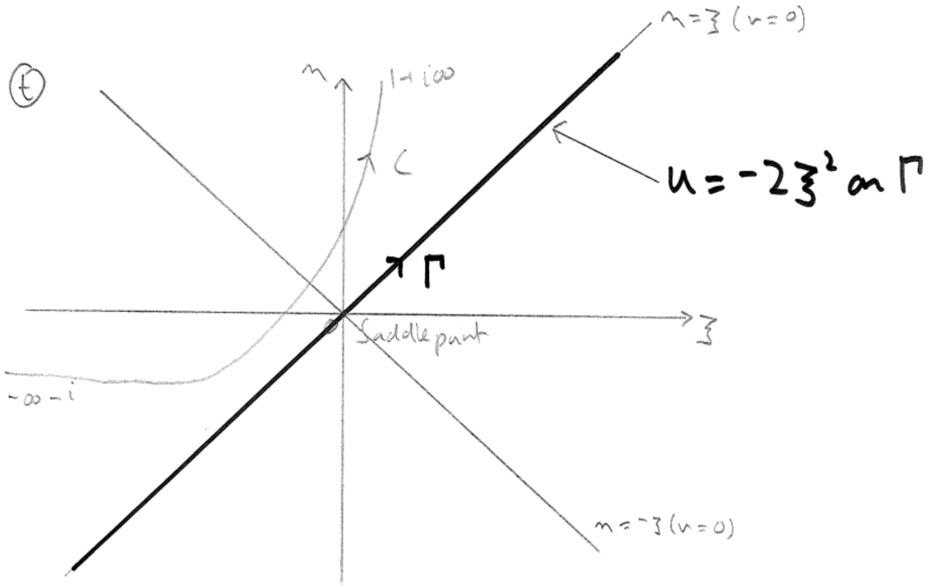
Arrows point in direction of steepest descent.

- C infinite with endpoints in different valleys

$\Rightarrow$ Γ must pass through a saddle point, with u decreasing with distance from it.

Example

$\phi = it^2$



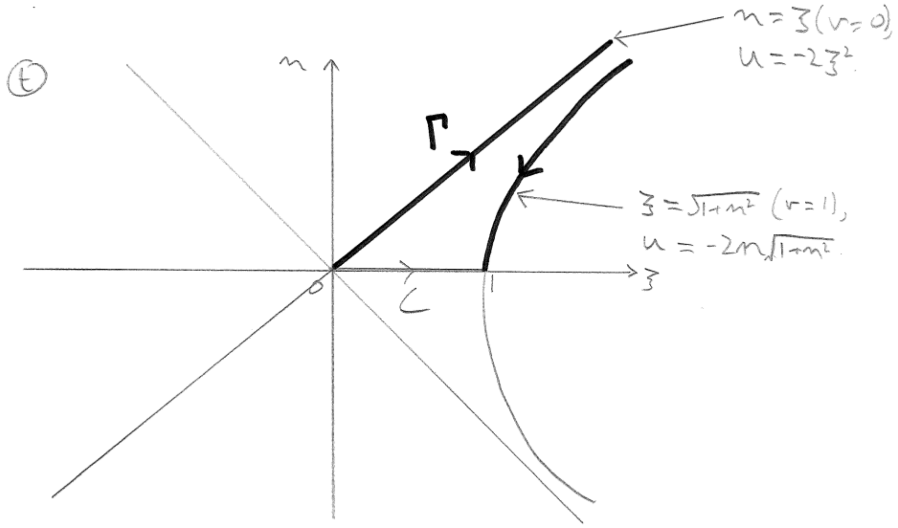
- Method therefore called "method of steepest descents" or "saddle point method".

- C finite and v different at endpoints

  $\Rightarrow$ steepest paths through endpoints can only meet at a singularity or branch point or ∞.

Example

$\phi = it^2$



- In <u>summary</u> the method is

  (1) deform contour to union of steepest descent contours through endpoints and any relevant saddle points;

  (2) evaluate local contributions from saddle and end points using Laplace's method.

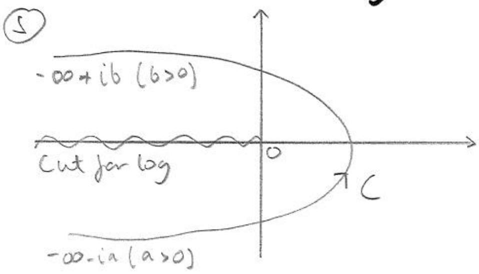
- When deforming contour from C to Γ we must include the contributions from any poles crossed and avoid crossing any branch cuts that have been used in the definition of $f(t)$ and $\phi(t)$.

Examples

① $I(x) = \int_0^1 e^{ixt^2}\,dt$ as $x \to \infty$.

② $I(x) = \int_C e^s s^{-x}\,ds$ as $x \to \infty$, where $s^{-x} := e^{-x \operatorname{Log} s}$,

we take principal branch of log and C as shown:



③ $Ai(\pm x) = \frac{1}{2\pi} \int_C e^{i(t^3/3 \pm xt)}$ as $x \to \infty$, where C starts at ∞ with $\frac{2\pi}{3} < \arg(t) < \pi$ and ends at ∞ with $0 < \arg(t) < \frac{\pi}{3}$.