

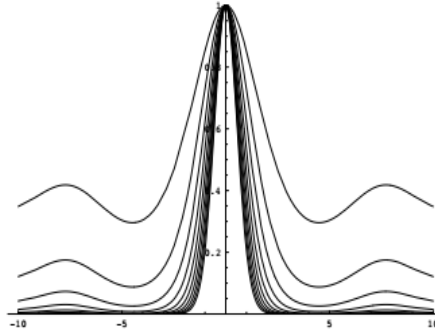
**PERTURBATION METHODS IN ENGINEERING
LECTURE 4**

LAPLACE'S METHOD AND THEIR APPLICATIONS WITH EXAMPLES

Consider the integral

$$I(x) = \int_a^b f(t)e^{x\phi(t)} dt.$$

We have seen that the dominant contribution to the integral will come from the place where $\phi(t)$ is largest.



There are three cases to consider

1. The maximum is at $t = a$.
2. The maximum is at $t = b$.
3. The maximum is at some $t = c$, with $a < c < b$.

In each case the argument is as follows:

1. The dominant contribution to the integral comes from the near the maximum of ϕ . We can reduce the range of integration to this local contribution introducing only exponentially small errors.
2. Near this point we can expand ϕ and f in Taylor series.
3. After rescaling the integration variable, we can replace the integration limits by ∞ introducing only exponentially small errors.

Case 1: The maximum is at $t = a$. First we can split the integral into a local and nonlocal part:

$$I(x) = \int_a^{a+\epsilon} f(t)e^{x\phi(t)} dt + \int_{a+\epsilon}^b f(t)e^{x\phi(t)} dt,$$

where $x^{-1} \ll \epsilon \ll x^{-1/2}$ (we will see where these restrictions come from soon). The second integral is exponentially small compared to the first, since it is $O(e^{x\phi(a+\epsilon)})$ and $\phi(a+\epsilon) \sim \phi(a) + \epsilon\phi'(a)$. Thus the second integral is $O(e^{x\epsilon\phi'(a)})$ times the first (which we will see is $O(e^{x\phi(a)})$). This is why we need $x\epsilon \gg 1$ (remember that $\phi'(a) < 0$ since ϕ is maximum at $t = a$).

In the first it is OK to expand $\phi(t)$ and $f(t)$ as an asymptotic series about $t = a$:

$$\phi(t) \sim \phi(a) + (t-a)\phi'(a) + \dots, \quad f(t) \sim f(a) + (t-a)f'(a) + \dots.$$

Then

$$I(x) \sim \int_a^{a+\epsilon} (f(a) + (t-a)f'(a) + \dots) e^{x(\phi(a) + (t-a)\phi'(a) + \frac{(t-a)^2}{2}\phi''(a) + \dots)} dt$$

Now we rescale the integration variable to remove the x from the exponential, *i.e.* we set $x(t-a) = s$.

Then

$$I(x) \sim \frac{e^{x\phi(a)}}{x} \int_0^{x\epsilon} \left(f(a) + \frac{s}{x}f'(a) + \dots \right) e^{s\phi'(a) + \frac{s^2}{2x}\phi''(a) + \dots} ds.$$

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Note that $\phi'(a) < 0$, since ϕ is maximum at a . Now we can expand $e^{\frac{s^2}{2x}\phi''(a)+\dots}$ as $x \rightarrow \infty$ as

$$1 + \frac{s^2}{2x}\phi''(a) + \dots$$

This is OK providing $(x\epsilon)^2/x \ll 1$ i.e. $\epsilon \ll x^{-1/2}$. This is where the other restriction on ϵ comes from. Keeping only the leading-order term we have

$$I(x) \sim \frac{f(a)e^{x\phi(a)}}{x} \int_0^{x\epsilon} e^{s\phi'(a)} ds.$$

Now we can replace the upper limit by infinity, introducing only exponentially small errors:

$$I(x) \sim \frac{f(a)e^{x\phi(a)}}{x} \int_0^\infty e^{s\phi'(a)} ds = -\frac{f(a)e^{x\phi(a)}}{x\phi'(a)}.$$

Case 2: The maximum is at $t = b$. A similar argument shows that

$$I(x) \sim \frac{f(b)e^{x\phi(b)}}{x\phi'(b)}.$$

Case 3: The maximum is at $t = c$, $a < c < b$. First we can split the integral into a local and nonlocal part:

$$I(x) = \int_a^{c-\epsilon} f(t)e^{x\phi(t)} dt + \int_{c-\epsilon}^{c+\epsilon} f(t)e^{x\phi(t)} dt + \int_{c+\epsilon}^b f(t)e^{x\phi(t)} dt,$$

where in this case we will see that we need $1/x^{1/2} \ll \epsilon \ll 1/x^{1/3}$ (we will see where these restrictions come from shortly). The first and last integrals are exponentially small compared to the second, since they are $O(e^{x\phi(c+\epsilon)})$. In this case $\phi(c+\epsilon) \sim \phi(c) + \frac{\epsilon^2}{2}\phi''(c)$ because ϕ has a maximum at the interior point $t = c$ so $\phi'(c) = 0$. This is why we need $x\epsilon^2 \gg 1$, i.e. $x^{-1/2} \ll \epsilon$.

In the second integral it is OK to expand $\phi(t)$ and $f(t)$ as an asymptotic series about $t = c$:

$$\phi(t) \sim \phi(c) + \frac{(t-c)^2}{2}\phi''(c) + \frac{(t-c)^3}{6}\phi'''(c) + \dots, \quad f(t) \sim f(c) + (t-c)f'(c) + \dots$$

Then

$$I(x) \sim \int_{c-\epsilon}^{c+\epsilon} (f(c) + (t-c)f'(c) + \dots) e^{x(\phi(c) + \frac{(t-c)^2}{2}\phi''(c) + \frac{(t-c)^3}{6}\phi'''(c) + \dots)} dt$$

Now we rescale the integration variable to remove the x from the exponential, i.e. we set $\sqrt{x}(t-c) = s$ (note the different scaling of the contributing region). Then

$$I(x) \sim \frac{e^{x\phi(c)}}{\sqrt{x}} \int_{-\sqrt{x}\epsilon}^{\sqrt{x}\epsilon} \left(f(c) + \frac{s}{x}f'(c) + \dots \right) e^{\frac{s^2}{2}\phi''(c) + \frac{s^3}{6\sqrt{x}}\phi'''(c) + \dots} ds.$$

Note that $\phi''(c) < 0$, since ϕ has a maximum at $t = c$. Now we can expand $e^{\frac{s^3}{6\sqrt{x}}\phi'''(c)+\dots}$ as $x \rightarrow \infty$ as

$$1 + \frac{s^3}{6\sqrt{x}}\phi'''(c) + \dots$$

This is OK providing $(x^{1/2}\epsilon)^3/x^{1/2} \ll 1$, i.e. $\epsilon \ll x^{-1/3}$. This is where the other restriction on ϵ comes from. Keeping only the leading-order term we have

$$I(x) \sim \frac{f(c)e^{x\phi(c)}}{\sqrt{x}} \int_{-\sqrt{x}\epsilon}^{\sqrt{x}\epsilon} e^{\frac{s^2}{2}\phi''(c)} ds.$$

Now we can replace the upper and lower limits by $\pm\infty$, introducing only exponentially small errors:

$$I(x) \sim \frac{f(c)e^{x\phi(c)}}{\sqrt{x}} \int_{-\infty}^{\infty} e^{\frac{s^2}{2}\phi''(c)} ds = \frac{\sqrt{2\pi} f(c)e^{x\phi(c)}}{\sqrt{-x\phi''(c)}}.$$

Method of stationary phase

The method of stationary phase is used for problems in which the exponent ϕ is not real but purely imaginary, say $\phi(t) = i\psi(t)$, where $\psi(t)$ is real.

$$I(x) = \int_a^b f(t)e^{ix\psi(t)} dt.$$

Riemann-Lebesgue lemma If $\int_a^b |f(t)| dt < \infty$ and $\psi(t)$ is continuously differentiable for $a \leq t \leq b$ and not constant on any subinterval in $a \leq t \leq b$, then

$$\int_a^b f(t)e^{ix\psi(t)} dt \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

Useful when using integration by parts.

Example

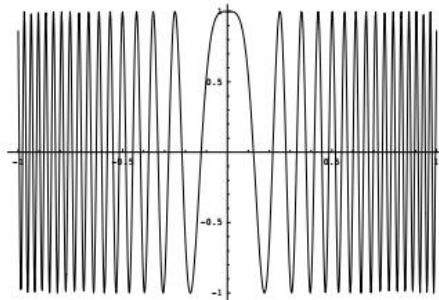
$$I(x) = \int_0^1 \frac{e^{ixt}}{1+t} dt.$$

Integrating by parts gives

$$I(x) = -\frac{ie^{ix}}{2x} + \frac{i}{x} - \frac{i}{x} \int_0^1 \frac{e^{ixt}}{(1+t)^2} dt.$$

The last integral is lower order by the Riemann-Lebesgue lemma.

Why is the Riemann-Lebesgue lemma true? Locally near any point $t = t_0$, $\psi(t) \sim \psi(t_0) + (t-t_0)\psi'(t_0) + \dots$ and the period of oscillation is $\frac{2\pi}{x\psi'(t_0)}$. As $x \rightarrow \infty$ this is very small, $f(t)$ is almost constant, and the contribution from the “up” and “down” parts of the oscillation almost cancel out. (You can find a rigorous proof of the Riemann-Lebesgue lemma in analysis books.) However, this is not true if $\psi'(t_0) = 0$. In this case the integrand oscillates much more slowly near t_0 , so that there is less cancellation. Here’s a plot of $\text{Re}(e^{100ix^2})$.



Suppose $\psi'(c) = 0$ with $a < c < b$, with $\psi'(t)$ being nonzero for $a \leq t < c$ and $c < t \leq b$. As for Laplace’s method, we split the range of integration

$$I(x) = \int_a^{c-\epsilon} f(t)e^{ix\psi(t)} dt + \int_{c-\epsilon}^{c+\epsilon} f(t)e^{ix\psi(t)} dt + \int_{c+\epsilon}^b f(t)e^{ix\psi(t)} dt,$$

where $\epsilon \ll 1$. The first and third integrals are lower order. To show this we use integration by parts

$$\begin{aligned} \int_a^{c-\epsilon} f(t)e^{ix\psi(t)} dt &= \int_a^{c-\epsilon} \frac{f(t)}{ix\psi'(t)} \frac{d}{dt} \left(e^{ix\psi(t)} \right) dt \\ &= \left[\frac{f(t)}{ix\psi'(t)} e^{ix\psi(t)} \right]_a^{c-\epsilon} - \frac{1}{x} \int_a^{c-\epsilon} e^{ix\psi(t)} \frac{d}{dt} \left(\frac{f(t)}{i\psi'(t)} \right) dt. \end{aligned}$$

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Providing the last integral exists it is lower order by the Riemann-Lebesgue lemma. The first integral is

$$O\left(\frac{1}{x\psi'(c-\epsilon)}\right) = O\left(\frac{1}{x\epsilon\psi''(c)}\right)$$

providing $\psi''(c) \neq 0$. For the second integral we expand ψ and f as an asymptotic series about $t = c$

$$f(t) \sim f(c) + (t-c)f'(c) + \dots, \quad \psi(t) \sim \psi(c) + \frac{(t-c)^2}{2}\psi''(c) + \frac{(t-c)^3}{6}\psi'''(c) + \dots$$

Then

$$\int_{c-\epsilon}^{c+\epsilon} f(t)e^{ix\psi(t)} dt \sim \int_{c-\epsilon}^{c+\epsilon} \left(f(c) + (t-c)f'(c) + \dots\right) e^{ix\left(\psi(c) + \frac{(t-c)^2}{2}\psi''(c) + \frac{(t-c)^3}{6}\psi'''(c) + \dots\right)} dt.$$

As for Laplace's method, we change the integration variable so that the oscillation is on an order one scale by setting $x^{1/2}(t-c) = s$ to give

$$\int_{c-\epsilon}^{c+\epsilon} f(t)e^{ix\psi(t)} dt \sim \frac{e^{ix\psi(c)}}{x^{1/2}} \int_{-x^{1/2}\epsilon}^{x^{1/2}\epsilon} \left(f(c) + \frac{s}{x^{1/2}}f'(c) + \dots\right) e^{i\frac{s^2}{2}\psi''(c) + i\frac{s^3}{6x^{1/2}}\psi'''(c) + \dots} ds.$$

Now we can expand $e^{i\frac{s^3}{6x^{1/2}}\psi'''(c) + \dots}$ as

$$1 + i\frac{s^3}{6x^{1/2}}\psi'''(c) + \dots$$

so long as $\epsilon \ll x^{-1/3}$. The leading order term is

$$\int_{c-\epsilon}^{c+\epsilon} f(t)e^{ix\psi(t)} dt \sim \frac{f(c)e^{ix\psi(c)}}{x^{1/2}} \int_{-x^{1/2}\epsilon}^{x^{1/2}\epsilon} e^{i\frac{s^2}{2}\psi''(c)} ds.$$

Now we replace the limits of integration by $\pm\infty$, which introduces error terms of order $1/(x\epsilon)$ (check by integration by parts). Hence

$$\int_{c-\epsilon}^{c+\epsilon} f(t)e^{ix\psi(t)} dt \sim \frac{f(c)e^{ix\psi(c)}}{x^{1/2}} \int_{-\infty}^{\infty} e^{i\frac{s^2}{2}\psi''(c)} ds + O\left(\frac{1}{x\epsilon}\right) = \frac{\sqrt{2\pi}f(c)e^{ix\psi(c)}e^{\pm i\pi/4}}{x^{1/2}|\psi''(c)|^{1/2}} + O\left(\frac{1}{x\epsilon}\right)$$

where (contour integration reveals that) the factor $e^{+i\pi/4}$ is used if $\psi''(c) > 0$ and $e^{-i\pi/4}$ is used if $\psi''(c) < 0$. Thus we need $x^{-1/2} \gg (\epsilon x)^{-1}$, i.e. $\epsilon \gg x^{-1/2}$, as in Laplace's method. The error is the same order as the neglected first and third integrals. So finally

$$I(x) = \frac{\sqrt{2\pi}f(c)e^{ix\psi(c)}e^{\pm i\pi/4}}{x^{1/2}|\psi''(c)|^{1/2}} + O\left(\frac{1}{x\epsilon}\right)$$

as $x \rightarrow \infty$ with $x^{-1/2} \ll \epsilon \ll x^{-1/3}$.

Important notes

- The error terms are only algebraically small, not exponentially small as in Laplace's method.
- Higher-order corrections are very hard to get since they may come from the whole range of integration. This is in contrast to Laplace's method where the full asymptotic expansion depends only on the local region because the errors are exponentially small.

Method of steepest descents

Laplace's method and the method of stationary phase are really just special cases of the general method of steepest descents, which is for integrals

$$I(x) = \int_C f(t)e^{x\phi(t)} dt,$$

where $f(t)$ and $\phi(t)$ are **complex**, and C is some contour in the complex t -plane.

We might expect, based on Laplace's method, that the important contribution to the integral as $x \rightarrow +\infty$ comes from the place where $\text{Re}(\phi)$ is maximum, at t_0 say, and that the integral is basically of size

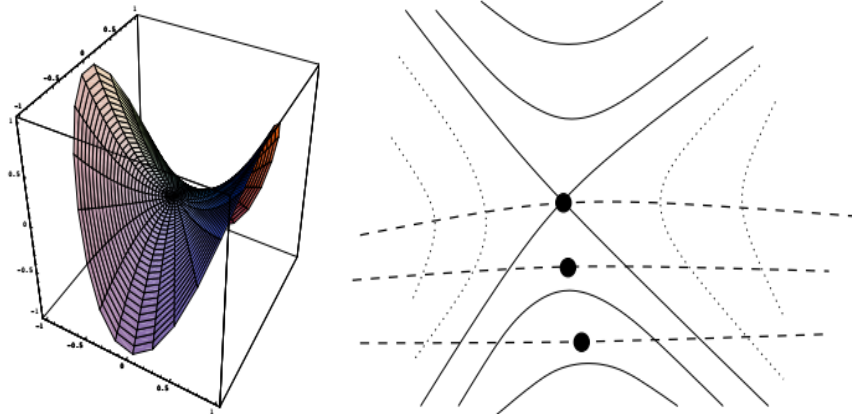
$$f(t_0)e^{x\phi(t_0)} \sqrt{\frac{2\pi}{-\lambda\phi''(t_0)}}$$

(where $'$ is the derivative along the path of integration). However, this estimate is **way too big**. The reason is that it ignores the rapid oscillation due to the imaginary part of ϕ , which causes cancellation exactly as in the method of stationary phase. We can see that the estimate above is wrong by deforming the contour a bit, which does not change the value of the integral, but which can change the maximum value of $\text{Re}(\phi)$.

Now, since $\phi(t) = u(\xi, \eta) + iv(\xi, \eta)$ is an analytic function of $t = \xi + i\eta$, we have the Cauchy-Riemann equations

$$u_\xi = v_\eta, \quad u_\eta = -v_\xi.$$

Hence $\nabla^2 u = u_{\xi\xi} + u_{\eta\eta} = 0$. This means that u cannot have any maxima or minima in the (ξ, η) -plane, only saddle points (since a maximum or minimum would require $u_{\xi\xi}u_{\eta\eta} > 0$). Thus the landscape of u has hills ($u > 0$) and valleys ($u < 0$) at infinity, with saddle points which are the passes from one valley into another. By the Cauchy-Riemann equations the saddle points are where $d\phi/dt = 0$. If our contour is infinite it must tend to infinity in a valley (see *e.g.* surface plot of $u(\xi, \eta) = \eta^2 - \xi^2$ for $\phi(t) = -t^2$). By deforming the contour we can keep reducing the maximum value of u , until the contour goes through the saddle point which is the lowest that u gets (see *e.g.* contour plot of u in which solid lines are for positive values of u , dotted lines are for negative values of u , and the dashed lines show C being deformed through the saddle point).



But why do we know that this is the right value. Suppose we can deform the contour C into one in which v is constant. Then there is no oscillation in the integrand, and the Laplace-type argument will work. Now if v is constant on the path, then $\nabla v = (v_\xi, v_\eta)$ is perpendicular to the path. By the Cauchy-Riemann equations this means that $\nabla u = (u_\xi, u_\eta)$ is parallel to the path, so that the path follows the steepest directions on the surface of u . There is only one path on which v is constant which goes to a valley at $\pm\infty$ and this is the path through the saddle. A little thought shows that this has to be the case. Since

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u is first increasing as we come up from one valley and then decreasing as we go off to another valley, we must go through a point where $du/ds = 0$, where s is distance along the path. Since v is constant, so that $dv/ds = 0$, everywhere on the path, we must go through a saddle point at which both $du/ds = 0$ and $dv/ds = 0$.

So the method of steepest descents is as follows:

- (i) Deform the contour to be the steepest descent contour through the relevant saddle point(s).
- (ii) Evaluate the local contribution from the saddle exactly as in Laplace's method.
- (iii) Evaluate the local contribution from the end point(s) exactly as in Laplace's method.

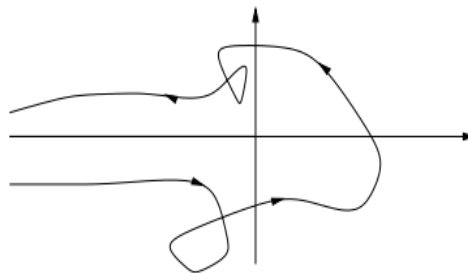
Remember that when deforming the contour we must include the contribution from any poles that we cross.

Of course, we could have chosen a path on which $u = \text{Re}(\phi)$ was constant and applied the method of stationary phase. However, we have seen that Laplace's method is far superior in that it can generate all the terms in the asymptotic series: the neglected "tails" of the integral are exponentially small. In fact, the best way to generate higher order terms in a stationary phase integral is to deform to the steepest descent contour.

Example: Steepest descents on the gamma function Consider as $x \rightarrow \infty$ the gamma function $\Gamma(x)$, which may be defined by

$$\frac{1}{\Gamma(x)} = \frac{1}{2\pi i} \int_{C'} e^{t-t^x} dt,$$

where C' is a contour which starts at $t = -\infty - ia$ ($a > 0$), encircles the branch cut that lies along the negative real axis, and then ends up at $t = -\infty + ib$ ($b > 0$).

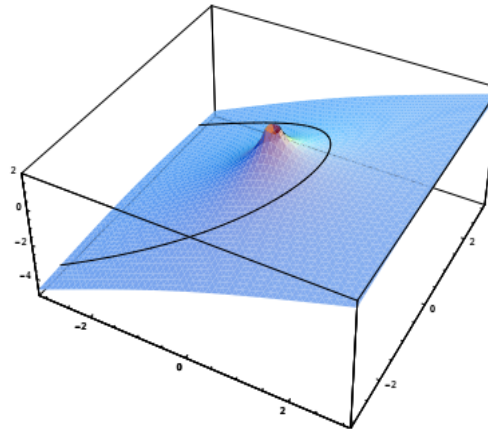
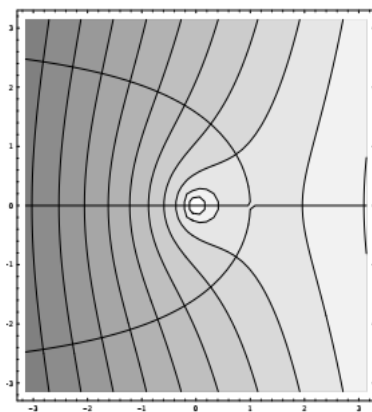


This is a moveable saddle problem. Writing $e^{t-t^x} = e^{t-x \ln t}$ and differentiating the whole exponent with respect to t shows that there is a saddle point at $t = x$. Thus we begin by changing the moveable saddle to a fixed saddle by the change of variable $t = xs$ to give

$$\frac{1}{\Gamma(x)} = \frac{1}{2\pi i x^{x-1}} \int_C e^{x(s-\log s)} ds = \frac{1}{2\pi i x^{x-1}} \int_C e^{x\phi(s)} ds$$

where $\phi = s - \log s$ and C is the rescaled contour (which we could take to be the same as C' by the deformation theorem). The saddle point(s) are now at $\phi'(s) = 0$, *i.e.* $s = 1$.

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We deform to the steepest descent contour $\text{Im}(s) = \arg(s)$ ($|\arg(s)| < \pi$) through the saddle as illustrated. Having deformed to the steepest descent contour the procedure is exactly that for Laplace's method. The integral is split into a local contribution from near the saddle and the rest, which is exponentially smaller. For the local contribution ϕ is expanded in a Taylor series about the saddle point $s = 1$ giving

$$\phi(s) \sim 1 + \frac{(s-1)^2}{2} - \frac{(s-1)^3}{3} + \dots$$

so that

$$\frac{1}{\Gamma(x)} \sim \frac{e^x}{2\pi i x^{x-1}} \int e^{\frac{x(s-1)^2}{2} - \frac{x(s-1)^3}{3} + \dots} ds.$$

At this stage the integral is from $-\epsilon$ to ϵ along the steepest descent contour from the saddle $s = 1$. We then rescale the integration variable so that the quadratic term in the exponent is $O(1)$ by setting $\sqrt{x}(s-1) = u$, giving

$$\frac{1}{\Gamma(x)} \sim \frac{e^x}{2\pi i x^{x-1} \sqrt{x}} \int e^{\frac{u^2}{2} - \frac{u^3}{3\sqrt{x}} + \dots} du,$$

where the integral is from $-x^{1/2}\epsilon$ to $x^{1/2}\epsilon$ along the steepest descent contour. We now expand $e^{-\frac{u^3}{3\sqrt{x}} + \dots}$ keeping only the first term and replace the integration limits by $\pm\infty$ along the steepest descent contour (introducing only exponentially small errors), giving

$$\frac{1}{\Gamma(x)} \sim \frac{e^x}{2\pi i x^{x-1/2}} \int e^{\frac{u^2}{2}} du.$$

Now the steepest descent contour is locally parallel to the imaginary axis near to the saddle point $s = 1$, so we set $u = iv$. A comparison with the figure above tells us which way to integrate – in this case from $v = -\infty$ to $v = \infty$. Thus,

$$\frac{1}{\Gamma(x)} \sim \frac{e^x}{2\pi x^{x-1/2}} \int_{-\infty}^{\infty} e^{-\frac{v^2}{2}} dv = \frac{e^x}{\sqrt{2\pi} x^{x-1/2}},$$

i.e.

$$\Gamma(x) \sim \sqrt{2\pi} x^{x-1/2} e^{-x} \quad \text{as } x \rightarrow \infty.$$

Example: Steepest descents on the Airy function

1. Positive argument Consider as $x \rightarrow \infty$ the Airy function

$$\text{Ai}(x) = \frac{1}{2\pi} \int_{C'} e^{i(t^3/3+xt)} dt,$$

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where C' is a contour that starts at infinity with $2\pi/3 < \arg(t) < \pi$ and ends at infinity with $0 < \arg(t) < \pi/3$. Note that the integrand decays at infinity where $\text{Re}(it^3) < 0$, *i.e.* in the sectors defined by $0 < \arg(t) < \pi/3$, $2\pi/3 < \arg(t) < \pi$ and $4\pi/3 < \arg(t) < 5\pi/3$.

This is a moveable saddle problem. Differentiating the whole exponent shows that the saddle points are at $t = \pm ix^{1/2}$. Thus we rescale $t = x^{1/2}z$ to give

$$\text{Ai}(x) = \frac{x^{1/2}}{2\pi} \int_C e^{ix^{3/2}(z^3/3+z)} dz = \frac{x^{1/2}}{2\pi} \int_C e^{x^{3/2}\phi(z)} dz,$$

where $\phi(z) = i(z^3/3 + z)$ and C is the rescaled contour, which we could take to be the same as C' by the deformation theorem and must start in the sector V_1 and end in the sector V_2 shown in figure 1(a).

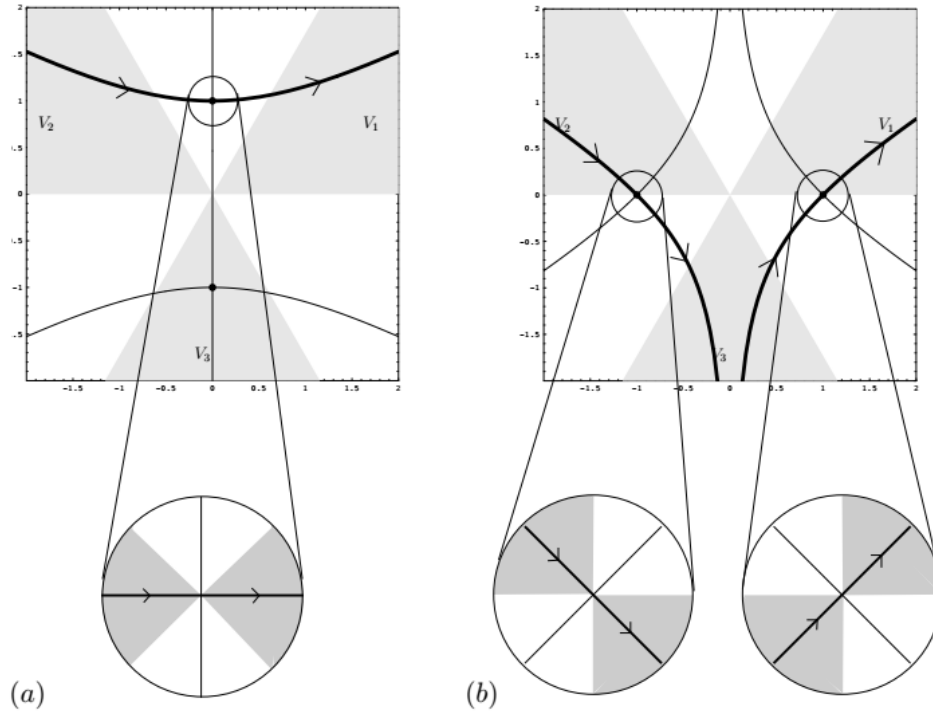


Figure 1: Steepest descent curves for (a) $x \rightarrow \infty$ and (b) $x \rightarrow -\infty$. Note that the shading shows the sectors which are valleys **at infinity** and is not supposed to be a contour plot of the magnitude of the integrand, but just an aid to determine the steepest descent (rather than ascent) contour.

The saddle points are the points where $\phi'(z) = 0$, *i.e.* $z = \pm i$. We deform the contour C to the steepest descent contour from V_2 to V_1 , which goes through the saddle point $z = i$ but not the saddle point $z = -i$.

Having deformed to the steepest descent contour the procedure is exactly that for Laplace's method. The integral is split into a local contribution from near the saddle and the rest, which is exponentially smaller. For the local contribution ϕ is expanded in a Taylor series about the saddle point $z = i$ as

$$\phi(z) \sim -\frac{2}{3} - (z-i)^2 + \dots,$$

so that

$$\text{Ai}(x) \sim \frac{x^{1/2}e^{-2x^{3/2}/3}}{2\pi} \int e^{-x^{3/2}(z-i)^2+\dots} dz.$$

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At this stage the integral is from $-\epsilon$ to ϵ along the steepest descent contour from the saddle $z = i$. Now we change variable by setting $x^{3/4}(z - i) = u$ to give

$$\text{Ai}(x) \sim \frac{e^{-2x^{3/2}/3}}{2\pi x^{1/4}} \int e^{-u^2 + \dots} du,$$

where the integral is from $-x^{3/4}\epsilon$ to $x^{3/4}\epsilon$ along the steepest descent contour. We now replace these limits by $\pm\infty$ along the steepest descent contour (introducing only exponentially small errors). Keeping only the leading order term we therefore have

$$\text{Ai}(x) \sim \frac{e^{-2x^{3/2}/3}}{2\pi x^{1/4}} \int e^{-u^2} du,$$

where the integral goes to infinity along the steepest descent contour. The steepest descent contour is given by $-u^2$ real and negative, *i.e.* u real. A comparison with figure 1(a) tells us which way to integrate – in this case from $-\infty$ to ∞ . Thus

$$\text{Ai}(x) \sim \frac{e^{-2x^{3/2}/3}}{2\pi x^{1/4}} \int_{-\infty}^{\infty} e^{-u^2} du = \frac{e^{-2x^{3/2}/3}}{2\sqrt{\pi} x^{1/4}}.$$