Course name: ***Algorithm of calculating methods***
Course language: **Uzbek**
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Lecture 6 : *Interpolation algorithm. Interpretation of functions.*

Fan nomi: ***Hisoblash usullarini algoritmlash***
Fan o'qituvchisi: **Muhtorjon Yusupov**

6-mavzu: *Interpolyatsiyalash usullarini algoritmlash.*

6-mavzu. Interpolyatsiyalash usullarini algoritmlash.

Reja:

1. Masalaning qo`yilishi.
2. Chekli ayirmalar va ularning xossalari.
3. N'yutonning 1- interpolyatsion formulasi.
4. N'yutonning 2- interpolyatsion formulasi.

Tayanch iboralar:

Interpolyatsiyalar, ayirma, chekli ayirma, yig'indi, n-tartibli ayirma, N'yutonning interpolyatsion formulalari, interpolyatsiya tugun, interpolyatsiya qadami, chiziqli, parabolik, analitik ko`rinish, qoldiq xad, orkaga qarab interpolyatsiyalash.

1. MASALANING QO`YILISHI

Aksariyat hisoblash usullari masalaning qo`yilishida katnashadigan funktsiyalarni unga biror muayyan ma`noda yaqin va tuzilishi soddarok bo`lgan funktsiyalarga almashtirish goyasiga asoslangan. Bu bobda funktsiyalarni yaqinlashtirish masalasining eng sodda va juda keng qo`llaniladigan qismi — funktsiyalarni interpolyatsiyalash ma-salasi kurib chikiladi.

Interpolyatsiya masalasining moxiyati quyidagidan iborat. Faraz kilaylik $u=f(x)$ funktsiya jadval ko`rinishida berilgan bo`lsin:

$$Y_0 = f(x_0),\ y_1 = f(x_1),\ldots,\ y_n = f(x_n)$$

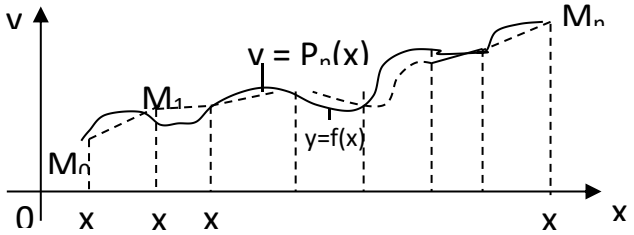
Odatda interpolyatsiyalash masalasi quyidagicha ko`rinishda qo`yiladi: Shundai n-tartiblidan oshmagan $R(x) = R_n(x)$ ko`pxad topish kerakki, $P(x_i)$ berilgan $x_i(i=0,1,1,\ldots,n)$ nuqtalarda $f(x)$ bilan bir xil qiymatlarni qabul kilsin, ya`ni $P(x_i) = y_i$.

Bu masalaning geometrik ma`nosi quyidagidan iborat:

darajasi p dan ortmaydigan shunday

$$y = P_n(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n \qquad (1)$$

ko`pxad kurilsinki, uning grafigi berilgan $M_i\ (x_i,\ u_i)$ $(i = 0,1,\ \ldots\ n)$ nuqtalardan utsin (9-rasm). Bu erdagi x_i $(i=0,1,2,\ldots n)$ nuqtalar interpolyatsiya tugun nuqtalari yoki tugunlar deyiladi. $R(x)$ esa interpolyatsiyalovchi funktsiya deyiladi.



9- rasm

Amalda topilgan $R(x)$ interpolyatsion formula $f(x)$ funktsiyaning berilgan x argumentning (interpolyatsiya tugunlaridan farqli) qiymatlarini hisoblash uchun qo`llaniladi. Ushbu operatsiya funktsiyani interpolyatsiyalash deyiladi. (Agar $x \in (a,b)$ bo`lsa interpolyatsiyalash $x \in (a,b)$ bo`lsa, ekstrapolyatsiyalash deyiladi).

CHEKLI AYIRMALAR VA ULARNING XOSSALARI

Faraz kilaylik argumentning o`zaro teng o`zoklikda joylashgan $x_i=x_0+ih$, $\Delta x_i = x_{i+1} - x_i = h = const$ (h-jadval kadami) qiymatlarida $f(x)$ funktsiyaning moc ravishdagi $y_i=f(x_i)$ qiymatlari berilgan bo`lsin.

Birinchi tartibli chekli ayirmalar deb

$$\Delta y_i=f(x_{i+1}) - f(x_i) = y_{i+1} - y_i \qquad (2)$$

ifodaga ikkinchi tartibli chekli ayirmalar deb

$$\Delta^2 y_i=\Delta(\Delta y_i) = \Delta y_{i+1} - \Delta y_i = y_{i+2} - 2y_{i+1} + y_i \qquad (3)$$

ifodaga va xokazo n-tartibli chekli ayirmalar deb

$$\Delta^n y_i = \Delta(\Delta^{n-1} y_i) = \Delta^{n-1} y_{i+1} - \Delta^{n-1} y_i \qquad (4)$$

ifodaga aytiladi. CHekli ayirmalarni quyidagi 1- jadval ko`rinishida kam olish mumkin.

1-jadval

x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$	...
x_0	y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$	
x_1	y_1	Δy_1	$\Delta^2 y_1$	$\Delta^3 y_0$		
x_2	y_2	Δy_2	$\Delta^2 y_2$			
x_3	y_3	Δy_3				
x_4	y_4					
…						

(2) dan quyidagiga egamiz

$$y_{i+1}= y_1+\Delta y_i = (1+\Delta) y_i \qquad (5)$$

Bu erdan ketma-ket quyidagilarni keltirib chikaramiz:

$$y_{i+2} = (1+\Delta)y_{i+1} = (1+\Delta)^2 y_i,$$
$$y_{i+3} = (1+\Delta)y_{i+2} = (1+\Delta)^3 y_i$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$y_{i+n} = (1+\Delta)^n y_i$$

N’yuton binomi formulasidan foydalanib, quyidagiga ega bo`lamiz:

$$y_{i+n} =y_i + C_n^1 \Delta y_i + \ldots + \Delta^n y_i$$

Bundan esa:

$$\Delta^n y_i = \left[(1+\Delta)-1\right]^n y_i = (1+\Delta)^n y_i - C_n^1 (1+\Delta)^{n-1} y_i + C_n^2 (1+\Delta)^{n-2} y_i - \ldots + (-1)^n y_i$$

yoki

$$\Delta^n y_i = y_{i+1} - C_n^1 y_{n+i-1} + C_n^2 y_{n+i-2} - ... + (-1)^n y_i \qquad (6)$$

Masalan, (6) dan

$$\Delta^2 y_i = y_{i+2} - 2y_{i+1} + y_i,$$

$$\Delta^3 y_i = y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_1$$

va x.k.

CHekli ayirmalar quyidagi x o s s a l a r g a ega.

1.Funktsiyalar yig'indisining (ayirmasining) chekli ayirmasi funktsiyalarning chekli anirmalari yig'indisiga (ayirmasiga) teng:

$$\Delta^n (f(x) \pm \varphi(x)) = \Delta^n f(x) \pm \Delta^n \varphi(x).$$

Funktsiya o`zgarmas songa ko`paytirilsa, uning chekli ayirmasi usha songa ko`payadi:

$$\Delta^n (k \cdot f(x)) = k \cdot \Delta^n f(x).$$

n-tartibli chekli ayirmaning /p-tartibli chekli ayirmasi *(p+t)- tartibli* chekli ayirmaga teng:

$$\Delta^m (\Delta^n y) = \Delta^{m+n} y.$$

n-tartibli ko`paddning p-tartibli chekli ayirmasi o`zgarmas songa, *n+1*-tartibli chekli ayirmasi esa nolga teng.

Misol. Jadval kadamini $h=1$ va dastlabki qiymatni $x_o = 0$ deb xisoblab, $u = 2x^3 - 2x^2 + Zx - 1$ ko`pxadning ayirmalar jadvali to`zilsin.

E c h i s h . u ning $x_0=0,\ x_1=1,\ x_2=2,\ x_3=3$ nuqtalardagi qiymatlarini hisoblaymiz: $y_0 = -1,\ y_1=2,\ y_2 = 13,\ y_3 = 4$ Bundan esa quyidagilar kelib chikadi: $\Delta y_0 = y_1 - y_0 = 3,\ \Delta y_1 = y_2 - y_1 = 11,\ \Delta^2 y_0 = \Delta y_1 - \Delta y_0 = 8.$ Bu qiymatlarni 2-jadvalga joylashtiramiz:

1-jadval

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	-1	8		
1	2	11	8	12
2	13	31	20	12
3	44	63	32	12
4	107	107	44	
5	214			
…	…	…	…	…

Berilgan funktsiya Z- darajali kundad bo`lganligi sa-babli uning 3-tartibli ayirmasi o`zgarmas son bo`lib, $\Delta^3 y = 12$ bo`ladi. Jadvalning kolgan ustunlari

$$\Delta^2 y_{i+1} = \Delta^2 y_i + 12, \qquad (i=0,1,2,...);$$

$$\Delta y_{i+1} = \Delta y_i + \Delta^2 y_i \qquad (i=1,2,...);$$

$$y_{i+1} = y_i + \Delta y_i \qquad (i=2,3,..)$$

formulalar yordamida to`ldiriladi.

N'YUTONNING 1- INTERPOLYATSION FORMULASI

Faraz kilaylik $y=f(x)$ funktsiya uchun $y_1=f(x)$ qiymatlar berilgan va interpolyatsiya tugunlari teng o`zoklikda joylashgan bo`lsin, ya`ni $x_1=x_0+ih$ $(I=0,1,2,...,h)$ ($h$ – interpolyatsiya kadami). Argumentning moc qiymatlarida darajasi h dan oshmaydigan moc qiymatlar oladigan ko`pxad tuzish lozim bo`lsin va bu ko`pxad kuiidagi ko`rinishga ega bo`lsin:

$$P_n(x) = a_0+a_1(x-x_0)+a_2(x-x_0)\ (x-x_1)+\ldots+a_n(x-x_0)\ (x-x_1)\ldots(x-x_{n-1}). \quad (7)$$

Bu n-tartibli ko`pxad. Interpolyatsiya masalasidagi shartga ko`ra $R(x)$ ko`pxad $x_0, x_1, \ldots, x_n$ interpolyatsiya tugunlarida $P_n(x_0)=y_0$. $P_n(x_1)=y_1$, $P_n(x_2)=y_2$,... $P_n(x_n)=y_n$ qiymatlarni qabul kiladi. $x=x_0$ deb tasavvur etsak, ( 7) formuladan $y_0=P_n(x_0)=a_0$, ya`ni $a_0=y_0$. So`ngra $x$ ga $x_1$ va $x_2$ larning qiymatlarini berib, ketma-ket quyidagiga ega bo`lamiz:

$$y_1=P_n(x_1)=a_0+a_1(x_1-x_0), \text{ bundan } a_1=\frac{\Delta y_0}{h}$$

$$y_2=P_n(x_2)=a_0+a_1(x_2-x)+a_2(x_2-x_0)(x_2-x_1),$$

ya`ni $y_2 - 2\Delta y_0 - y_0=2h^2a_2$

yoki $y_2-2y_1+y_0=2h^2a_2$, bundan $a_2=\frac{\Delta^2 y_0}{2!h^2}$

Bu jarayonni davom ettirib, $x=x_n$ uchun kuiidagi ifodani hosil kilamiz:

$$a_n=\frac{\Delta^n y_0}{n!h^n}$$

Topilgan $a_0, a_1, a_2,\ldots a_n$ koeffitsientlarning qiymatlarini (7) formulaga kuysak,

$$P_n(x)=y_0+\frac{\Delta y_0}{1!h}(x-x_0)+\frac{\Delta^2 y_0}{2!h^2}(x-x_0)(x-x_1)+\ldots+\frac{\Delta^n y_0}{n!y^n}(x-x_0)\cdot\ldots\cdot(x-x_{n-1})$$

(8)

ko`rinishga ega bo`lamiz. Bu formulada $\frac{x-x_0}{h}=q$, ya`ni $x = x_0+hq$ belgilash kiritilsa, u xolda

$$\frac{x-x_1}{h}=\frac{x-x_0-h}{h}=q-1,$$

$$\frac{x-x_2}{h}=\frac{x-x_0-2h}{h}=q-2, \text{ va x.k.}$$

Natijada N'yutonning 1- interpolyatsion formulasiga ega bo`lamiz:

$$P_n(x)=P_n(x_0+qh)=y_0+q\Delta y_0+\frac{q(q-1)}{2!}\Delta^2 y_0+\ldots$$
$$+\frac{q(q-1)\cdot\ldots\cdot(q-n+1)}{n!}\Delta^n y_0 \qquad (9)$$

N'yutonning 1-interpolyatsion formulasini [*a, b*] ning boshlangich nuqtalarida qo`llash qulay.

Agar $p==1$ bo`lsa, u xolda $P_1(x)=y_0+q\,\Delta y_0$ ko`rinishdagi chiziqli interpolyatsion formulaga, $p=2$ bo`lganda esa

$$P_2(x)=y_0+q\Delta y_0+\frac{q(q-1)}{2}\Delta^2 y_0$$

ko`rinishdagi parabolik interpolyatsion formulaga ega bo`lamiz.

N'yutonning 1-formulasini *oldinga qarab interpolyatsiyalash formulasi ham* deyiladi.

(9) formulaning koldik, xadi

$$P_n(x)=h^{n+1}\frac{q(q-1)\ldots(q-n)}{(n+1)!}f^{(n+1)}(\xi) \qquad (10)$$

bu erda $\xi\in[x_0, x_n]$

Funktsiyaning analitik ko`rinishi har doim ham ma`lum bulavermaydi. Bundai xollarda chekli ayirmalar to`zilib,

$$f^{(n+1)}(\xi)\approx\frac{\Delta^{n+1}y_0}{h^{n+1}}$$

deb olinadi. U xolda N'yutonning birinchi interpolyatsion formulasi uchun xatolik

$$P_n(x)\approx\frac{q(q-1)\ldots(q-n)}{(n+1)!}\Delta^{(n+1)}y_0 \qquad (11)$$

formula orqali topiladi.

Misol. *u=lgx* funktsiyaning 3-jadvalda berilgan qiymatlaridan foydalanib, uning x=1001 bo`lgan xoldagi qiymatini toping.

3-jadval

x	*y*	*Δy*	*Δ²y*	*Δ³y*
1000	3,0000000	43214	- 426	8
1010	3,0043214	42788	- 418	9
1020	3,0086002	42370	- 409	8
1030	3,0128372	41961	- 401	
1040	3,0170333	41560		
1050	3,0211893			

E c h i s h . CHekli aiirmalar jadvalini to`zamiz. 3- jadvaldan ko`rinib turibdiki, 3-tartibli chekli ayirma o`zgarmas, shu sababli (9) formula uchun n=3 olish etarli:

$$y(x)=P_3(x)=y_0+q\Delta y_0+\frac{q(q-1)}{2!}\Delta^2 y_0+\frac{q(q-1)(q-2)}{3!}\Delta^3 y_0$$

x=1001 uchun *q = 0,1* (*h=10*). Shuning uchun

$$\lg 1001=3{,}0000000+0{,}1\cdot 0{,}0043214+\frac{0{,}1\cdot 0{,}9}{2}\times$$

$$\times 0{,}0000426+\frac{0{,}1\cdot 0{,}9\cdot 1{,}9}{6}\cdot 0{,}0000008=3{,}0004341$$

Endi koldik xadni baxolaymiz. (10) formulaga asosan n=3 bo`lganda quyidagiga egamiz:

$$R_3(x)=\frac{h^4\cdot q(q-1)(q-2)(q-3)}{4!}f^{(4)}(\xi)$$

bu erda 1000<ξ<1030.

f(x) =lgx bo`lgani sababli $f^{(4)}(x)=-\frac{3!}{x^4}\lg e$; shuning uchun

$$\left|f^{(4)}(\xi)\right|<\frac{3!}{(1000)^4}\lg e.$$

h=10 va *q=0,1* uchun quyidagiga ega bo`lamiz:

$$\left|R_3(1001)\right|=\frac{0{,}1\cdot 0{,}9\cdot 1{,}9\cdot 2{,}9\cdot 10^4\lg e}{4\cdot(1000)^4}\approx 0{,}5\cdot 10^{-9}$$

Shunday kilib, koldik xad $R_3(1001)\approx 0{,}5\cdot 10^{-9}$ ekan.

N’YUTONNING 2 - INTERPOLYATSION FORMULASI

N’yutonning birinchi interpolyatsion formulasi jadvalning boshida va ikkinchi formulasi esa jadvalning oxirida interpolyatsiyalash uchun muljallangan. N’yutonning ikkinchi interpolyatsion formula-sini keltirib chikaramiz.

Faraz kilaylik *y=f(x)* funktsiyaning *n+1* ta qiymati ma`lum bo`lsin; ya`ni argumentning *n=1* ta $x_0, x_1, x_2, \ldots, x_n$ qiymatlarida funktsiyaning qiymatlar $y_0, y_1, y_2, \ldots, y_n$ bo`lsin. Tugunlar orasidagi masofa *h* o`zgarmas bo`lsin. Quyidagi ko`rinishdagi interpolyatsion ko`pxadni ko`ramiz:

$$P_n(x)=a_0+a_1(x-x_n)+a_2(x-x_n)(x-x_n-1)+a_3(x-x_n)(x-x_n-1)(x-x_n-2)+\ldots+$$
$$+a_n(x-x_n)(x-x_{n-1})\ldots(x-x_1) \qquad (12)$$

Bunda katnashayotgan $a_0, a_1, \ldots, a_n$ noma`lum koeffitsientlarni topishni $x=x_n$ bo`lgan xoldan boshlash kerak. So`ngra argumentga $x_{n-1}, x_{n-2}, \ldots$ qiymatlar berib, kolgan koeffitsientlar animanadi.

Yuqorida kurilgan muloxazalarni (12) formula uchun ham qo`llasak, u xolda noma`lum koeffitsientlar $a_1, a_2, a_3, \ldots, a_n$ larni topish uchun quyidagilarni hosil kilamiz:

$$a_0 = y_n, \qquad a_1 = \frac{\Delta y_{n-1}}{1!h}, \quad a_2 = \frac{\Delta^2 y_{n-2}}{2!h^2}, \qquad ..., a_n = \frac{\Delta_n y_0}{n!h^n}$$

Topilgan koeffitsientlarning qiymatlarini (12) formulaga kuysak,

$$P_n(x) = y_n + \frac{\Delta y_{n-1}}{1!h}(x - x_n) + \frac{\Delta^2 y_{n-2}}{2!h^2}(x - x_n)x - x_{(n-1)} + ... + \frac{\Delta^n y_0}{n!h^n}(x - x_n)...(x - x_1)$$

(13)

ko`rinishdagi *N'yutonning ikkinchi interpolyatsion formulasi* kelib chikadi. Bu formulada $q=(x-x_n)/h$ belgilash kiritsak,

$$P_n(x) = y_n + q\Delta y_{n-1} + \frac{q(q+1)}{2!}\Delta^2 y_{n-2} + ... + \frac{q(q+1)...(q+n-1)}{n!}\Delta^n y_0 \qquad (14)$$

hosil bo`ladi. Ba`zan bu formulani *orkaga qarab interpolyatsiyalash formulasi* ham deyiladi. (14) formuladan $[a,b]$ kesmaning oxirgi nuqtalarida foydalanish qulayrokdir.

N'yutonning ikkinchi interpolyatsion formulasining koldik xadini baxolash formulasi quyidagicha bo`ladi:

$$P_n(x) = h^{n+1}\frac{q(q+1)...(q+n)}{(n+1)!}F^{(n+1)}(\xi)$$

bu erda $q=(x-x_n)/h$, $\xi \in [x_0, x_n]$

Agar funktsiyaning analitik ko`rinishi ma`lum bo`lmasa, u xolda chekli ayirmalar to`zilib,

$$f^{(n+1)}(\xi) \approx \frac{\Delta^{n+1} y_0}{h^{n+1}}$$

deb olinadi. Shuning uchun N'yutonning ikkinchi interpolyatsion formulasi uchun xatolik formulasi

$$P_n(x) \approx \frac{q(q+1)...(q+n)}{(n+1)!}\Delta^{(n+1)} y_n$$

bo`ladi.

Misol. $u=lgx$ funktsiyaning 4-jadvalda bermlgan qiymatlaridan foydalanib, uning x=1044 dagi qiymatini hisoblang (h=10).

4- jadval

X	y
1000	3,0000000
1010	3,0043214
1020	3,0086002
1030	3,0128372
1040	3,0170333
1050	3,0211893

E c h i s h . CHekli ayirmalar jadvalini to`zamiz:

-jadval

x	U	Δu	Δ^2u	Δ^3u
1000	3,0000000	73214	- 426	8
1010	3,0043214	42788	- 418	9
1020	3,0086002	42370	- 409	8
1030	3,0128372	41961	- 401	
1040	3,0170333	41560		
1050	3,0211893			

$x_n = 1050$ bo`lsin, u xolda

$$q = \frac{x - x_n}{h} = \frac{1044 - 1050}{10} = -0.6$$

5- jadvaldagi tagiga chizilgan ayirmalardan foydalangan xolda (14) formulaga asosan quyidagiga ega bo`lamiz:

$$\lg 1044 = 3{,}0211893 + (-0{,}6)\cdot 0{,}0041560 + \frac{(-0{,}6)\cdot(-0{,}6+1)}{2} \times$$

$$\times 0{,}0000401 + \frac{(-0{,}6)\cdot(-0{,}6+1)\cdot(-0{,}6+2)}{6}\cdot 0{,}0000008 = 3{,}0187005$$

Takrorlash uchun savollar:

1. Odatda interpolyatsiya masalasi qanday ko`rinishda qo`yiladi?
2. Interpolyatsiya masalasining geometrik ma`nosi qanday ko`rinishdan iborat?
3. Interpolyatsiya tugun nuqtalari deb kaysi nuqtalarga aytiladi?
4. Interpolyatsiyalovchi funktsiya deb qanday funktsiyalarga aytiladi?
5. 1-tartibli chekli ayirmalar deb qanday ifodaga aytiladi?
6. N’yuton-Binomi formulasini keltiring.
7. N’yutonning 1-interpolyatsion formulasini yozing.
8. N’yutonning 2-interpolyatsion formulasini yozing.
9. Orkaga qarab interpolyatsiyalash formulasini yozing.