ASYMPTOTIC APPROXIMATION OF INTEGRALS

Integration by parts

We have already seen the use of integration by parts to obtain an asymptotic approximation of the error function. Here we show some more examples.

Example 1. Derivation of an asymptotic power series If $f(\epsilon)$ is differentiable near $\epsilon = 0$ then local behaviour of $f(\epsilon)$ near 0 may be studied using integration by parts. We write

$$f(\epsilon) = f(0) + \int_0^\epsilon f'(x)\,dx.$$

Integrating by parts once gives

$$f(\epsilon) = f(0) + \left[(x-\epsilon)f'(x)\right]_0^\epsilon + \int_0^\epsilon (\epsilon - x)f''(x)\,dx.$$

Repeating $N-1$ times gives

$$f(\epsilon) = \sum_{n=0}^{N} \frac{\epsilon^n f^{(n)}(0)}{n!} + \frac{1}{N!}\int_0^\epsilon (\epsilon - x)^N f^{(N+1)}(x)\,dx.$$

If the remainder term exists for all N and sufficiently small $\epsilon > 0$ then

$$f(\epsilon) \sim \sum_{n=0}^{\infty} \frac{\epsilon^n f^{(n)}(0)}{n!} \qquad \text{as } \epsilon \to 0.$$

If the series converges then it is just the Taylor expansion of $f(\epsilon)$ about $\epsilon = 0$.

Example 2.

$$I(x) = \int_x^\infty e^{-t^4}\,dt.$$

As $x \to \infty$,

$$\begin{aligned} I(x) &= -\frac{1}{4}\int_x^\infty \frac{1}{t^3}\frac{d}{dt}(e^{-t^4})\,dt \\ &= \left[-\frac{e^{-t^4}}{4t^3}\right]_x^\infty - \frac{3}{4}\int_x^\infty \frac{1}{t^4}e^{-t^4}\,dt \\ &= \frac{e^{-x^4}}{4x^3} - \frac{3}{4}\int_x^\infty \frac{1}{t^4}e^{-t^4}\,dt. \end{aligned}$$

The first term is the leading-order asymptotic approximation because

$$\int_x^\infty \frac{1}{t^4}e^{-t^4}\,dt < \frac{1}{x^4}\int_x^\infty e^{-t^4}\,dt = \frac{1}{x^4}I(x) \ll I(x) \qquad \text{as } x \to \infty.$$

Further integration by parts gives more terms in the asymptotic series.

Example 3.

$$I(x) = \int_0^x t^{-1/2}e^{-t}\,dt.$$

Here we need to be more careful because the naive approach

$$I(x) = \left[-t^{-1/2}e^{-t}\right]_0^x - \frac{1}{2}\int_0^x t^{-3/2}e^{-t}\,dt$$

gives $\infty - \infty$. Instead we express $I(x)$ as the difference between two integrals

$$I(x) = \int_0^\infty t^{-1/2}e^{-t}\,dt - \int_x^\infty t^{-1/2}e^{-t}\,dt.$$

The first integral is finite, independent of x; it has the value $\Gamma(1/2) = \sqrt{\pi}$. The second may be integrated by parts successfully, because the contribution from the endpoint vanishes.

$$\begin{aligned}\int_0^x t^{-1/2}e^{-t}\,dt &= \sqrt{\pi} + \int_x^\infty t^{-1/2}\frac{d}{dt}(e^{-t})\,dt \\ &= \sqrt{\pi} - \frac{e^{-x}}{\sqrt{x}} + \frac{1}{2}\int_x^\infty t^{-3/2}e^{-t}\,dt.\end{aligned}$$

General rule: Integration by parts will not work if the contribution from one of the limits of integration is much larger than the size of the integral. Here $I(x)$ is finite for all $x > 0$, but at the endpoint $t = 0$ the integrand has a singularity, which gets worse on differentiating.

Failure of integration by parts

$$I(x) = \int_0^\infty e^{-xt^2}\,dt.$$

If we try integration by parts we find

$$\int_0^\infty e^{-xt^2}\,dt = \int_0^\infty \left(-\frac{1}{2xt}\right)\left(-2xte^{-xt^2}\right)dt = \left[\frac{e^{-xt^2}}{-2xt}\right]_0^\infty - \int_0^\infty \frac{1}{2xt^2}e^{-xt^2}\,dt.$$

The final integral does not exist, a sure sign that integration by parts has failed. In fact, $I(x)$ has the exact value $\sqrt{\pi}/(2\sqrt{x})$. Integration by parts could never pick up this fractional power, and is doomed to failure. Integration by parts will also not work when the dominant contribution to the integral comes from an interior point rather than an end point. While integration by parts is simple to use and gives an explicit error term that can often be rigorously bounded, it is of limited applicability and inflexible.

4.3 Laplace's method

Laplace's method is a general technique for obtaining the behaviour as $x \to +\infty$ of integrals of the form

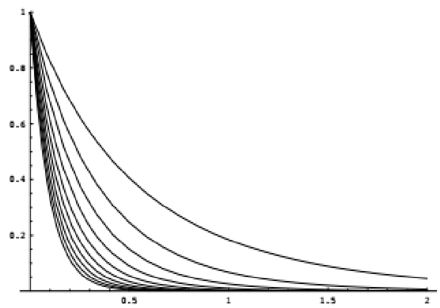
$$I(x) = \int_a^b f(t)e^{x\phi(t)}\,dt,$$

where $f(t)$ and $\phi(t)$ are real continuous functions.

Example Find the asymptotic behaviour of

$$I(x) = \int_0^{10} \frac{e^{-xt}}{(1+t)}\,dt$$

as $x \to +\infty$. The integrand is shown for $x = 1, \cdots, 10$.



As $x \to \infty$ the largest contribution to the integral comes from near $t = 0$ because this is where $-t$ is biggest. For values of t away from zero the integrand is exponentially small. So split the range of integration:

$$I(x) = \int_0^{\epsilon} \frac{e^{-xt}}{(1+t)}\,dt + \int_{\epsilon}^{10} \frac{e^{-xt}}{(1+t)}\,dt$$

where $x^{-1} \ll \epsilon \ll 1$. The second integral is $O(e^{-\epsilon x})$ which is exponentially small by comparison to the first, so we can neglect it. In the first integral t is small so we can Taylor expand $1/(1+t)$. The best way to be systematic is to change variable $xt = s$, giving

$$I(x) \sim \frac{1}{x}\int_0^{x\epsilon} \frac{e^{-s}}{(1+s/x)}\,ds.$$

Since $x\epsilon$ (the largest value of s) is $\ll x$ we Taylor expand $1/(1+s/x)$ to give

$$I(x) \sim \frac{1}{x}\int_0^{x\epsilon} e^{-s}\sum_{n=0}^{\infty}\frac{(-s)^n}{x^n}\,ds = \sum_{n=0}^{\infty}\frac{1}{x^{n+1}}\int_0^{x\epsilon}(-s)^n e^{-s}\,ds,$$

since the expansion is uniform on $0 < s < \epsilon x$. Finally, we can now replace the upper limit $x\epsilon$ by infinity in each sum, introducing only exponentially small errors again because integration by parts shows that

$$\int_{x\epsilon}^{\infty} s^n e^{-s}\,ds = O((x\epsilon)^n e^{-\epsilon x}).$$

Hence

$$I(x) \sim \sum_{n=0}^{\infty}\frac{1}{x^{n+1}}\int_0^{\infty}(-s)^n e^{-s}\,ds = \sum_{n=0}^{\infty}\frac{(-1)^n n!}{x^{n+1}} \qquad \text{as } x \to \infty.$$

4.4 Watson's lemma

The method of the example can be justified using Watson's lemma, which applies to integrals of the form

$$I(x) = \int_0^b f(t)e^{-xt}\,dt, \qquad b > 0.$$

Suppose $f(t)$ is continuous on the interval $0 \le t \le b$ and has the asymptotic series expansion

$$f(t) \sim t^{\alpha}\sum_{n=0}^{\infty} a_n t^{\beta n} \qquad \text{as } t \to 0+,$$

where $\alpha > -1$ and $\beta > 0$ so that the integral converges at $t = 0$. If $b = \infty$ it is also necessary that $f(t) \ll e^{ct}$ as $t \to +\infty$ for some positive constant c so that the integral converges at $t = \infty$. Then Watson's lemma states that

$$I(x) \sim \sum_{n=0}^{\infty}\frac{a_n\Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}} \qquad \text{as } x \to +\infty.$$

The derivation of Watson's Lemma is basically by the same method as in the example if the asymptotic series for f is uniformly convergent in a neighbourhood of the origin (as is often the case in practice). If this is not the case (as it is in general), then it is no longer possible to interchange the order of integration and summation: we work instead with a finite number of terms in the asymptotic expansion of f by writing, for each positive integer N,

$$f(t) = t^{\alpha}\sum_{n=0}^{N-1} a_n t^{\beta n} + \mathrm{O}(t^{\beta N}) \qquad \text{as } t \to 0+;$$

the result is then readily derived by showing that, for each positive integer N,

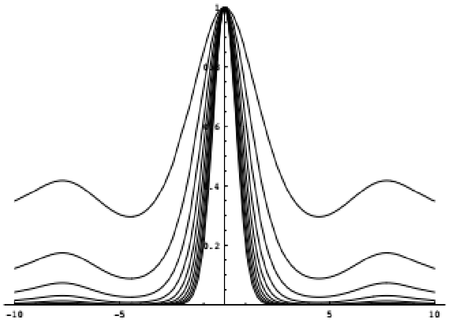
$$I(x) = \sum_{n=0}^{N-1}\frac{a_n\Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}} + \mathrm{O}\left(\frac{1}{x^{\alpha+\beta N+1}}\right) \qquad \text{as } x \to +\infty.$$

Asymptotic expansion of general Laplace integrals

Consider the integral

$$I(x) = \int_a^b f(t)e^{x\phi(t)}\,dt.$$

We have seen that the dominant contribution to the integral will come from the place where $\phi(t)$ is largest.



There are three cases to consider

1. The maximum is at $t = a$.
2. The maximum is at $t = b$.
3. The maximum is at some $t = c$, with $a < c < b$.

In each case the argument is as follows:

1. The dominant contribution to the integral comes from the near the maximum of ϕ. We can reduce the range of integration to this local contribution introducing only exponentially small errors.
2. Near this point we can expand ϕ and f in Taylor series.
3. After rescaling the integration variable, we can replace the integration limits by ∞ introducing only exponentially small errors.

Case 1: The maximum is at $t = a$. First we can split the integral into a local and nonlocal part:

$$I(x) = \int_a^{a+\epsilon} f(t)e^{x\phi(t)}\,dt + \int_{a+\epsilon}^b f(t)e^{x\phi(t)}\,dt,$$

where $x^{-1} \ll \epsilon \ll x^{-1/2}$ (we will see where these restrictions come from soon). The second integral is exponentially small compared to the first, since it is $O(e^{x\phi(a+\epsilon)})$ and $\phi(a+\epsilon) \sim \phi(a) + \epsilon\phi'(a)$. Thus the second integral is $O(e^{x\epsilon\phi'(a)})$ times the first (which we will see is $O(e^{x\phi(a)})$). This is why we need $x\epsilon \gg 1$ (remember that $\phi'(a) < 0$ since ϕ is maximum at $t = a$).

In the first it is OK to expand $\phi(t)$ and $f(t)$ as an asymptoptic series about $t = a$:

$$\phi(t) \sim \phi(a) + (t-a)\phi'(a) + \cdots, \qquad f(t) \sim f(a) + (t-a)f'(a) + \cdots.$$

Then

$$I(x) \sim \int_a^{a+\epsilon} \left(f(a) + (t-a)f'(a) + \cdots\right) e^{x(\phi(a)+(t-a)\phi'(a)+\frac{(t-a)^2}{2}\phi''(a)+\cdots)}\,dt$$

Now we rescale the integration variable to remove the x from the exponential, *i.e.* we set $x(t-a) = s$. Then

$$I(x) \sim \frac{e^{x\phi(a)}}{x}\int_0^{x\epsilon} \left(f(a) + \frac{s}{x}f'(a) + \cdots\right) e^{s\phi'(a)+\frac{s^2}{2x}\phi''(a)+\cdots}\,ds.$$

Note that $\phi'(a) < 0$, since ϕ is maximum at a. Now we can expand $e^{\frac{s^2}{2x}\phi''(a)+\cdots}$ as $x \to \infty$ as

$$1 + \frac{s^2}{2x}\phi''(a) + \cdots.$$

This is OK providing $(x\epsilon)^2/x \ll 1$ *i.e.* $\epsilon \ll x^{-1/2}$. This is where the other restriction on ϵ comes from. Keeping only the leading-order term we have

$$I(x) \sim \frac{f(a)e^{x\phi(a)}}{x}\int_0^{x\epsilon} e^{s\phi'(a)}\,ds.$$

Now we can replace the upper limit by infinity, introducing only exponentially small errors:

$$I(x) \sim \frac{f(a)e^{x\phi(a)}}{x}\int_0^{\infty} e^{s\phi'(a)}\,ds = -\frac{f(a)e^{x\phi(a)}}{x\phi'(a)}.$$

Case 2: The maximum is at $t = b$. A similar argument shows that

$$I(x) \sim \frac{f(b)e^{x\phi(b)}}{x\phi'(b)}.$$

Case 3: The maximum is at $t = c$, $a < c < b$. First we can split the integral into a local and nonlocal part:

$$I(x) = \int_a^{c-\epsilon} f(t)e^{x\phi(t)}\,dt + \int_{c-\epsilon}^{c+\epsilon} f(t)e^{x\phi(t)}\,dt + \int_{c+\epsilon}^{b} f(t)e^{x\phi(t)}\,dt,$$

where in this case we will see that we need $1/x^{1/2} \ll \epsilon \ll 1/x^{1/3}$ (we will see where these restrictions come from shortly). The first and last integrals are exponentially small compared to the second, since they are $O(e^{x\phi(c+\epsilon)})$. In this case $\phi(c+\epsilon) \sim \phi(c) + \frac{\epsilon^2}{2}\phi''(c)$ because ϕ has a maximum at the interior point $t = c$ so $\phi'(c) = 0$. This is why we need $x\epsilon^2 \gg 1$, *i.e.* $x^{-1/2} \ll \epsilon$.

In the second integral it is OK to expand $\phi(t)$ and $f(t)$ as an asymptoptic series about $t = c$:

$$\phi(t) \sim \phi(c) + \frac{(t-c)^2}{2}\phi''(c) + \frac{(t-c)^3}{6}\phi'''(c)\cdots, \qquad f(t) \sim f(c) + (t-c)f'(c) + \cdots.$$

Then

$$I(x) \sim \int_{c-\epsilon}^{c+\epsilon} \left(f(c) + (t-c)f'(c) + \cdots\right) e^{x\left(\phi(c) + \frac{(t-c)^2}{2}\phi''(c) + \frac{(t-c)^3}{6}\phi'''(c)+\cdots\right)}\,dt$$

Now we rescale the integration variable to remove the x from the exponential, *i.e.* we set $\sqrt{x}\,(t-c) = s$ (note the different scaling of the contributing region). Then

$$I(x) \sim \frac{e^{x\phi(c)}}{\sqrt{x}}\int_{-\sqrt{x}\,\epsilon}^{\sqrt{x}\,\epsilon} \left(f(c) + \frac{s}{x}f'(c) + \cdots\right) e^{\frac{s^2}{2}\phi''(c) + \frac{s^3}{6\sqrt{x}}\phi'''(c)+\cdots}\,ds.$$

Note that $\phi''(c) < 0$, since ϕ has a maximum at $t = c$. Now we can expand $e^{\frac{s^3}{6\sqrt{x}}\phi'''(c)+\cdots}$ as $x \to \infty$ as

$$1 + \frac{s^3}{6\sqrt{x}}\phi'''(c) + \cdots.$$

This is OK providing $(x^{1/2}\epsilon)^3/x^{1/2} \ll 1$, *i.e.* $\epsilon \ll x^{-1/3}$. This is where the other restriction on ϵ comes from. Keeping only the leading-order term we have

$$I(x) \sim \frac{f(c)e^{x\phi(c)}}{\sqrt{x}}\int_{-\sqrt{x}\,\epsilon}^{\sqrt{x}\,\epsilon} e^{\frac{s^2}{2}\phi''(c)}\,ds.$$

Now we can replace the upper and lower limits by $\pm\infty$, introducing only exponentially small errors:

$$I(x) \sim \frac{f(c)e^{x\phi(c)}}{\sqrt{x}}\int_{-\infty}^{\infty} e^{\frac{s^2}{2}\phi''(c)}\,ds = \frac{\sqrt{2\pi}\,f(c)e^{x\phi(c)}}{\sqrt{-x\phi''(c)}}.$$

Method of stationary phase

The method of stationary phase is used for problems in which the exponent ϕ is not real but purely imaginary, say $\phi(t) = i\psi(t)$, where $\psi(t)$ is real.

$$I(x) = \int_a^b f(t)e^{ix\psi(t)}\,dt.$$

Riemann-Lebesgue lemma If $\int_a^b |f(t)|\,dt < \infty$ and $\psi(t)$ is continuously differentiable for $a \le t \le b$ and not constant on any subinterval in $a \le t \le b$, then

$$\int_a^b f(t)e^{ix\psi(t)}\,dt \to 0 \quad \text{as } x \to \infty.$$

Useful when using integration by parts.

Example

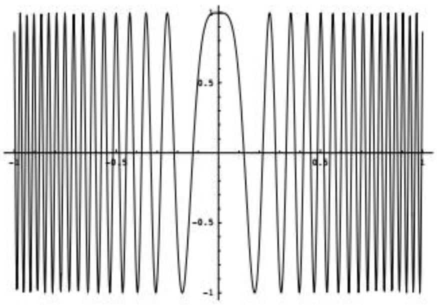
$$I(x) = \int_0^1 \frac{e^{ixt}}{1+t}\,dt.$$

Integrating by parts gives

$$I(x) = -\frac{ie^{ix}}{2x} + \frac{i}{x} - \frac{i}{x}\int_0^1 \frac{e^{ixt}}{(1+t)^2}\,dt.$$

The last integral is lower order by the Riemann-Lebesgue lemma.

Why is the Riemann-Lebesgue lemma true? Locally near any point $t = t_0$, $\psi(t) \sim \psi(t_0)+(t-t_0)\psi'(t_0)+\cdots$ and the period of oscillation is $\frac{2\pi}{x\psi'(t_0)}$. As $x \to \infty$ this is very small, $f(t)$ is almost constant, and the contribution from the "up" and "down" parts of the oscillation almost cancel out. (You can find a rigorous proof of the Riemann-Lebesgue lemma in analysis books.) However, this is not true if $\psi'(t_0) = 0$. In this case the integrand oscillates much more slowly near t_0, so that there is less cancellation. Here's a plot of $\mathrm{Re}(e^{100ix^2})$.



Suppose $\psi'(c) = 0$ with $a < c < b$, with $\psi'(t)$ being nonzero for $a \le t < c$ and $c < t \le b$. As for Laplace's method, we split the range of integration

$$I(x) = \int_a^{c-\epsilon} f(t)e^{ix\psi(t)}\,dt + \int_{c-\epsilon}^{c+\epsilon} f(t)e^{ix\psi(t)}\,dt + \int_{c+\epsilon}^{b} f(t)e^{ix\psi(t)}\,dt,$$

where $\epsilon \ll 1$. The first and third integrals are lower order. To show this we use integration by parts

$$\begin{aligned}\int_a^{c-\epsilon} f(t)e^{ix\psi(t)}\,dt &= \int_a^{c-\epsilon} \frac{f(t)}{ix\psi'(t)}\frac{d}{dt}\left(e^{ix\psi(t)}\right)\,dt \\ &= \left[\frac{f(t)}{ix\psi'(t)}e^{ix\psi(t)}\right]_a^{c-\epsilon} - \frac{1}{x}\int_a^{c-\epsilon} e^{ix\psi(t)}\frac{d}{dt}\left(\frac{f(t)}{i\psi'(t)}\right)\,dt.\end{aligned}$$

Providing the last integral exists it is lower order by the Riemann-Lebesgue lemma. The first integal is

$$O\left(\frac{1}{x\psi'(c-\epsilon)}\right) = O\left(\frac{1}{x\epsilon\psi''(c)}\right)$$

providing $\psi''(c) \neq 0$. For the second integral we expand ψ and f as an asymptoptic series about $t = c$

$$f(t) \sim f(c) + (t-c)f'(c) + \cdots, \qquad \psi(t) \sim \psi(c) + \frac{(t-c)^2}{2}\psi''(c) + \frac{(t-c)^3}{6}\psi'''(c) + \cdots.$$

Then

$$\int_{c-\epsilon}^{c+\epsilon} f(t)e^{ix\psi(t)}\,dt \sim \int_{c-\epsilon}^{c+\epsilon} \left(f(c) + (t-c)f'(c) + \cdots\right) e^{ix\left(\psi(c) + \frac{(t-c)^2}{2}\psi''(c) + \frac{(t-c)^3}{6}\psi'''(c) + \cdots\right)}\,dt.$$

As for Laplace's method, we change the integration variable so that the oscillation is on an order one scale by setting $x^{1/2}(t-c) = s$ to give

$$\int_{c-\epsilon}^{c+\epsilon} f(t)e^{ix\psi(t)}\,dt \sim \frac{e^{ix\psi(c)}}{x^{1/2}} \int_{-x^{1/2}\epsilon}^{x^{1/2}\epsilon} \left(f(c) + \frac{s}{x^{1/2}}f'(c) + \cdots\right) e^{i\frac{s^2}{2}\psi''(c) + i\frac{s^3}{6x^{1/2}}\psi'''(c) + \cdots}\,ds.$$

Now we can expand $e^{i\frac{s^3}{6x^{1/2}}\psi'''(c)+\cdots}$ as

$$1 + i\frac{s^3}{6x^{1/2}}\psi'''(c) + \cdots$$

so long as $\epsilon \ll x^{-1/3}$. The leading order term is

$$\int_{c-\epsilon}^{c+\epsilon} f(t)e^{ix\psi(t)}\,dt \sim \frac{f(c)e^{ix\psi(c)}}{x^{1/2}} \int_{-x^{1/2}\epsilon}^{x^{1/2}\epsilon} e^{i\frac{s^2}{2}\psi''(c)}\,ds.$$

Now we replace the limits of integration by $\pm\infty$, which introduces error terms of order $1/(x\epsilon)$ (check by integration by parts). Hence

$$\int_{c-\epsilon}^{c+\epsilon} f(t)e^{ix\psi(t)}\,dt \sim \frac{f(c)e^{ix\psi(c)}}{x^{1/2}} \int_{-\infty}^{\infty} e^{i\frac{s^2}{2}\psi''(c)}\,ds + O\left(\frac{1}{x\epsilon}\right) = \frac{\sqrt{2\pi}f(c)e^{ix\psi(c)}e^{\pm i\pi/4}}{x^{1/2}|\psi''(c)|^{1/2}} + O\left(\frac{1}{x\epsilon}\right)$$

where (contour integration reveals that) the factor $e^{+i\pi/4}$ is used if $\psi''(c) > 0$ and $e^{-i\pi/4}$ is used if $\psi''(c) < 0$. Thus we need $x^{-1/2} \gg (\epsilon x)^{-1}$, *i.e.* $\epsilon \gg x^{-1/2}$, as in Laplace's method. The error is the same order as the neglected first and third integrals. So finally

$$I(x) = \frac{\sqrt{2\pi}f(c)e^{ix\psi(c)}e^{\pm i\pi/4}}{x^{1/2}|\psi''(c)|^{1/2}} + O\left(\frac{1}{x\epsilon}\right)$$

as $x \to \infty$ with $x^{-1/2} \ll \epsilon \ll x^{-1/3}$.