ASYMPTOTIC APPROXIMATIONS AND ITS PROPERTIES

3.1 Definitions

Convergence

- A series $\sum_{n=0}^{\infty} f_n(z)$ converges at fixed z if $\forall \varepsilon > 0$, $\exists N_0(\varepsilon, z) \in \mathbb{N}$ s.t. $\left| \sum_{n=M}^{N} f_n(z) \right| < \varepsilon \quad \forall N \geq M > N_0$.
- It converges to $f(z)$ if $\forall \varepsilon > 0$ $\exists N_1(\varepsilon, z) \in \mathbb{N}$ s.t. $\left| \sum_{n=0}^{N} f_n(z) - f(z) \right| < \varepsilon \ \forall N > N_1$.
- A series converges if its terms decay sufficiently rapidly as $n \to \infty$.
- Less useful in practice than might be believed!

Example: $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$, $z \in \mathbb{C}$.

- Since e^{-t^2} is a holomorphic function of t on $\mathbb{C}$, it can be expanded in a Taylor series $\sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!}$, which has ∞ radius of convergence.
- Hence, can integrate term by term to obtain

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)\, n!}$$

$$= \frac{2}{\sqrt{\pi}} \left(z - \frac{z^3}{3} + \frac{z^5}{10} + \cdots \right),$$

which also has ∞ radius of convergence.

- For accuracy of 10^{-5}, 8 terms needed for $z=1$
  16 " " " $z=2$
  31 " " " $z=3$
  75 " " " $z=5$!
- Some terms very large (e.g. for $z=5$, largest term is $\approx 6.6\times 10^{8}$), so round-off error will swamp the answer for z sufficiently large.
- Problem is that approx. does not get better with each successive term until we have a lot of terms.

- An alternative approx. to erf(z) can be found by writing

$$\operatorname{erf}(z) = 1 - \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-t^2}\,dt$$

and integrating by parts (IBP):

$$\int_z^\infty e^{-t^2}\,dt = \int_z^\infty \underbrace{\left(-\frac{1}{2t}\right)}_{f}\underbrace{\left(-2te^{-t^2}\right)}_{g'}\,dt$$

$$= f(t)g(t)\Big|_z^\infty - \int_z^\infty f'(t)\,g(t)\,dt$$

$$= \frac{1}{2z}e^{-z^2} - \int_z^\infty \frac{1}{2t^2}e^{-t^2}\,dt.$$

- Continuing IBPs gives

$$\operatorname{erf}(z) = 1 - \frac{e^{-z^2}}{\sqrt{\pi}\,z}\left(1 - \frac{1}{2z^2} + \frac{1\cdot 3}{(2z^2)^2} - \frac{1\cdot 3\cdot 5}{(2z^2)^3} + \dots\right)$$

- This series diverges $\forall z \in \mathbb{C}$, but <u>truncated</u> series is very useful, e.g. for accuracy of 10^{-5}, only two terms are needed when $z = 3$!
- Importantly, for this series, the leading term is almost correct and addition of each extra term gets us closer to the answer, i.e. each of the corrections is of decreasing size, until they finally start to diverge.
- This is an <u>asymptotic</u> series.

Asymptoticness

- $\{f_n(\varepsilon)\}_{n \in \mathbb{N}_0}$ asymptotic if $\forall n \geq 1$ $\frac{f_n(\varepsilon)}{f_{n-1}(\varepsilon)} \to 0$ as $\varepsilon \to 0$.
- $f(\varepsilon) \sim \sum_{n=0}^{\infty} f_n(\varepsilon)$ as $\varepsilon \to 0$ if $\forall N \geq 0$ $\dfrac{f(\varepsilon) - \sum_{n=0}^{N} f_n(\varepsilon)}{f_N(\varepsilon)} \to 0$ as $\varepsilon \to 0$,

i.e. remainder smaller than last term included for ε sufficiently small.

- Usually leading or first few terms are sufficient for a good approximation.
- Often $f_n(\varepsilon) = a_n \varepsilon^n$, with $a_n \in \mathbb{R}$ independant of ε, in which case the asymptotic expansion

$$f(\varepsilon) \sim \sum_{n=0}^{\infty} a_n \varepsilon^n \quad \text{as } \varepsilon \to 0$$

is called an asymptotic power series.

Order notation

- $f(\varepsilon) = O(g(\varepsilon))$ as $\varepsilon \to \varepsilon_0$ means $\exists K, \delta > 0$ s.t. $|f(\varepsilon)| < K|g(\varepsilon)|$ $\forall |\varepsilon - \varepsilon_0| < \delta$.

- $f(\varepsilon) = o(g(\varepsilon))$ as $\varepsilon \to \varepsilon_0$ means $\dfrac{f(\varepsilon)}{g(\varepsilon)} \to 0$ as $\varepsilon \to \varepsilon_0$.

- $f(\varepsilon) = \mathrm{ord}(g(\varepsilon))$ as $\varepsilon \to \varepsilon_0$ means $\exists K \in \mathbb{R} \setminus \{0\}$ s.t. $\dfrac{f(\varepsilon)}{g(\varepsilon)} \to K$ as $\varepsilon \to \varepsilon_0$.

- NB: "O" does not imply "ord", but common to write "O" instead of "ord" when meaning is clear from context; this is because order is usually taken to be smallest function that works.

- E.g. $\sin x = O(1), o(1), O(x), \mathrm{ord}(x)$ as $x \to 0$;
  $\sin x = O(1)$ as $x \to \infty$;
  $\ln x = o(x^{-\delta})$ as $x \to 0^+$ $\forall \delta > 0$.

3.2 Uniqueness and manipulation of asymptotic series

- If a function $f(\varepsilon) \sim \sum_{n=0}^{\infty} a_n \delta_n(\varepsilon)$ as $\varepsilon \to 0$, then induction implies that $\{a_n\}_{n \in \mathbb{N}_0}$ is uniquely determined by

$$a_k = \lim_{\varepsilon \to 0} \frac{f(\varepsilon) - \sum_{n=0}^{k-1} a_n \delta_n(\varepsilon)}{\delta_k(\varepsilon)}.$$

- Uniqueness for a given sequence $\{\delta_n(\varepsilon)\}_{n \in \mathbb{N}_0}$, which may not be unique, e.g.

$$\tan \varepsilon \sim \varepsilon + \frac{\varepsilon^3}{3} + \frac{2\varepsilon^5}{15} + \ldots$$

$$\tan \varepsilon \sim \sin \varepsilon + \tfrac{1}{2}(\sin \varepsilon)^3 + \tfrac{3}{8}(\sin \varepsilon)^5 + \ldots \quad \text{as } \varepsilon \to 0.$$

- Uniqueness for a given function: two functions may share the same asymptotic expansion (a.e.) because they differ by a quantity smaller than the last term included, e.g.

$$e^{\varepsilon} \sim \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \quad \text{as } \varepsilon \to 0,$$

$$e^{\varepsilon} + e^{-1/\varepsilon^2} \sim \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \quad \text{as } \varepsilon \to 0.$$

- Two functions sharing the same asymptotic power series can only differ by a quantity that is not holomorphic, because two holomorphic functions with the same power series are identical.

- Asymptotic expansions can be naively added, subtracted, multiplied or divided, resulting in the correct asymptotic expansion for the sum, difference, product or quotient, perhaps based on an enlarged asymptotic sequence.

- This justifies expansion method for algebraic equations.

- One series can be substituted into another provided exponents of exponentials are calculated to ord(1), e.g. $f(z) = e^{z^2}$, $z(\varepsilon) = \varepsilon^{-1} + \varepsilon \Rightarrow$

$$f(z(\varepsilon)) = e^{(\frac{1}{\varepsilon}+\varepsilon)^2} = e^{1/\varepsilon^2} e^2 e^{\varepsilon^2} \sim e^{1/\varepsilon^2+2} \sum_{n=0}^{\infty} \frac{\varepsilon^{2n}}{n!} \quad \text{as } \varepsilon \to 0.$$

- sin and cos are exponentials.

- Asymptotic expansions can be integrated term by term w.r.t. ε resulting in the correct a.e. of the integral.

- In general asymptotic expansions cannot be differentiated safely, e.g.

$$f(\varepsilon) = \varepsilon\cos(1/\varepsilon) \qquad = O(\varepsilon) \quad \text{as } \varepsilon \to 0,$$
$$f'(\varepsilon) = \frac{1}{\varepsilon}\sin\left(\frac{1}{\varepsilon}\right) + \cos\left(\frac{1}{\varepsilon}\right) = O(1/\varepsilon) \quad \text{as } \varepsilon \to 0.$$

We would expect differentiating $O(\varepsilon)$ would give $O(1)$, but we got $O(1/\varepsilon)$.

- Terms move higher up expansion when integrated, but lower down when differentiated $\Rightarrow$ neglected h.o.t. may become impotant upon differentiating.

3.4 Parametric expansions

- Integrals and differential equations involve functions with one or more variables, e.g. $f(x;\varepsilon)$, with ε being a small parameter.

- $\exists$ obvious generalisation of definition of an a.e. by allowing the coefficients to depend on x: for fixed x,

$$f(x;\varepsilon) \sim \sum_{n=0}^{\infty} a_n(x)\,\delta_n(\varepsilon) \quad \text{as } \varepsilon \to 0$$

$$\Leftrightarrow \quad f(x;\varepsilon) - \sum_{n=0}^{N} a_n(x)\,\delta_n(\varepsilon) = o(\delta_N(\varepsilon)) \text{ as } \varepsilon \to 0 \ \forall N \in \mathbb{N}_0.$$