PERTURBATION METHODS IN ENGINEERING

LECTURE 1

INTRODUCTION TO REGULAR AND SINGULAR PERTURBATION THEORY.

Algebraic equations

Suppose we want to solve

$$x^2 + \epsilon x - 1 = 0$$

for x, where ϵ is a small parameter. The exact solutions are

$$x = -\frac{\epsilon}{2} \pm \sqrt{1 + \frac{\epsilon^2}{4}},$$

which we can expand using the binomial theorem:

$$x = \begin{cases} +1 - \frac{\epsilon}{2} + \frac{\epsilon^2}{8} - \frac{\epsilon^4}{128} + \cdots \\ -1 - \frac{\epsilon}{2} - \frac{\epsilon^2}{8} + \frac{\epsilon^4}{128} + \cdots \end{cases}$$

These expansions converge if $|\epsilon| < 2$. More important is that the truncated expansions give a good approximation to the roots when ϵ is small. For example, when $\epsilon = 0.1$:

x	$\sim$	1.0	1 term
		0.95	2 terms
		0.95125	3 terms
		0.951249	4 terms
exact	$=$	0.95124922...	

Here, we first found the exact solution, then approximated. Usually we need to make the approximation first, and then solve.

Iterative method

First, rearrange the equation so that it is in a form which can form the basis of an iterative process:

$$x = \pm\sqrt{1 - \epsilon x}.$$

Now, if we have an approximation to the positive root, x_n, say, a better approximation is given by

$$x_{n+1} = \sqrt{1 - \epsilon x_n}.$$

We need a starting point for the iteration: the solution when $\epsilon = 0$, $x_0 = 1$. After one iteration (on the positive root) we have

$$x_1 = \sqrt{1 - \epsilon}.$$

If we expand this as a binomial series we find

$$x_1 = 1 - \frac{\epsilon}{2} - \frac{\epsilon^2}{8} - \frac{\epsilon^3}{16} + \cdots .$$

We see that this is correct up to ϵ, but the ϵ^2 terms and higher are wrong. Hence we only need keep the first two terms

$$x_1 = 1 - \frac{\epsilon}{2} + \cdots .$$

Using this in the next iteration we have

$$x_2 = \sqrt{1 - \epsilon\left(1 - \frac{\epsilon}{2}\right)},$$

which can again be expanded to give

$$\begin{aligned} x_2 &= 1 - \frac{\epsilon}{2}\left(1 - \frac{\epsilon}{2}\right) - \frac{\epsilon^2}{8}\left(1 - \frac{\epsilon}{2}\right)^2 - \frac{\epsilon^3}{16}\left(1 - \frac{\epsilon}{2}\right)^3 + \cdots \\ &= 1 - \frac{\epsilon}{2} + \frac{\epsilon^2}{8} + \frac{\epsilon^3}{16} + \cdots . \end{aligned}$$

Now the ϵ^2 term is right, but the ϵ^3 term is still wrong. At each iteration more terms are correct, but more and more work is required. We can only check that a term is correct (without the exact solution) by proceeding to one more iteration and seeing if it changes.

The usual procedure is to place the dominant term of the equation on the x_{n+1} side (*i.e.*, the side that will give the new value), so that it can be calculated as a function of the terms on the x_n side (*i.e.*, the previously-obtained value). As we will see later, the identity of the dominant term can be adjusted by scaling.

Expansion method

First set $\epsilon = 0$ and find the unperturbed roots $x = \pm 1$ as in the iterative method. Now pose an expansion about one of these roots:

$$x = 1 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots .$$

Substitute the expansion into the equation:

$$(1 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots)^2 + \epsilon(1 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots) - 1 = 0.$$

Expanding the first term

$$1 + 2x_1\epsilon + (x_1^2 + 2x_2)\epsilon^2 + (2x_1x_2 + 2x_3)\epsilon^3 + \cdots + \epsilon + \epsilon^2 x_1 + \epsilon^3 x_2 + \cdots - 1 = 0.$$

Now we equate coefficients of powers of ϵ.

At ϵ^0: $\quad 1 - 1 = 0.$

This level is automatically satisfied because we started the expansion with the correct value $x = 1$ at $\epsilon = 0$.

At ϵ^1: $\quad 2x_1 + 1 = 0, \quad$ *i.e.* $x_1 = -\frac{1}{2}.$

At ϵ^2: $\quad x_1^2 + 2x_2 + x_1 = 0, \quad$ *i.e.* $x_2 = \frac{1}{8},$

where the previously determined value of x_1 is used.

At ϵ^3: $\quad 2x_1x_2 + 2x_3 + x_2 = 0, \quad$ *i.e.* $x_3 = 0.$

The expansion method is much easier that the iterative method when working to higher orders. However, it is necessary to assume the form of the expansion (in powers of ϵ).

Singular perturbations

Consider the problem:

$$\epsilon x^2 + x - 1 = 0.$$

When $\epsilon = 0$ there is just one root $x = 1$, but when $\epsilon \neq 0$ there are two roots. This is an example of a **singular perturbation** problem, in which the limit problem $\epsilon = 0$ differs in an important way from the limit $\epsilon \to 0$. The most interesting problems are often singular. Problems which are not singular are said to be **regular**.

To see what is happening let us look at the exact solutions

$$x = \frac{-1 \pm \sqrt{1+4\epsilon}}{2\epsilon},$$

and expand them for small ϵ (convergent if $|\epsilon| < 1/4$). The expansions of the two roots are

$$x = \begin{cases} 1 - \epsilon + 2\epsilon^2 - 5\epsilon^4 + \cdots \\ -\frac{1}{\epsilon} - 1 + \epsilon - 2\epsilon^2 + 5\epsilon^4 + \cdots \end{cases}$$

Thus the second root disappears to $x = -\infty$ as $\epsilon \to 0$.

We see that to capture the second root we need to start the expansion not with ϵ^0 but with ϵ^{-1}:

$$x = \frac{x_{-1}}{\epsilon} + x_0 + \epsilon x_1 + \cdots .$$

Substituting into the equation gives

$$\epsilon\left(\frac{x_{-1}}{\epsilon} + x_0 + \epsilon x_1 + \cdots\right)^2 + \left(\frac{x_{-1}}{\epsilon} + x_0 + \epsilon x_1 + \cdots\right) - 1 = 0.$$

Expanding the first term gives

$$\frac{1}{\epsilon}x_{-1}^2 + 2x_{-1}x_0 + \epsilon(2x_{-1}x_1 + x_0^2) + \cdots + \frac{1}{\epsilon}x_{-1} + x_0 + \epsilon x_1 + \cdots - 1 = 0.$$

Comparing coefficients of ϵ^n gives

At ϵ^{-1}: $\quad x_{-1}^2 + x_{-1} = 0, \quad$ *i.e.* $x_{-1} = -1$ or 0.

The root $x_{-1} = 0$ leads to the regular root, so we consider the singular root $x_{-1} = -1$.

At ϵ^0: $\quad 2x_{-1}x_0 + x_0 - 1 = 0, \quad$ *i.e.* $x_0 = -1$.

At ϵ^1: $\quad 2x_{-1}x_1 + x_0^2 + x_1 = 0, \quad$ *i.e.* $x_1 = 1$.

Rescaling the equation

Instead of starting the expansion with ϵ^{-1}, a very useful idea for singular problems is to rescale the variables before making the expansion. If we introduce the rescaling

$$x = \frac{X}{\epsilon}$$

into the originally singular equation we find that the equation for X,

$$X^2 + X - \epsilon = 0,$$

is regular. Thus the problem of finding the correct starting point for the expansion can be viewed as the problem of finding a suitable rescaling to regularise the singular problem.

Finding the right rescaling

Systematic approach: general rescaling

First pose a general rescaling with scaling factor $\delta(\epsilon)$:

$$x = \delta X,$$

in which X is strictly of order one as $\epsilon \to 0$. This gives

$$\epsilon\delta^2 X^2 + \delta X - 1 = 0.$$

Then consider the dominant balance in the equation as δ varies from very small to very large.

(i) $\delta \ll 1$. Then

$$\epsilon\delta^2 X^2 + \delta X - 1 = \text{small} + \text{small} - 1,$$

which cannot possibly balance the zero on the right-hand side of the equation. As δ is gradually increased the first term to break the domination of the -1 term is δX, which comes into play when $\delta = 1$.

(ii) $\delta = 1$. Now the left-hand side is

$$\epsilon\delta^2 X^2 + \delta X - 1 = \text{small} + X - 1.$$

This can balance the zero on the right-hand side, and produces the regular root $X = +1+$ small.

(iii) $1 \ll \delta \ll \epsilon^{-1}$. Now the term δX dominates the left-hand side, since upon dividing by δ,

$$\frac{\epsilon\delta^2 X^2 + \delta X - 1}{\delta} = \text{small} + X + \text{small}.$$

This can only balance the zero on the right-hand side if $X = 0$, but that violates the restriction that X is strictly of order one. As δ is further increased the dominance of δX is broken when the first term comes into play at $\delta = \epsilon^{-1}$.

(iv) $\delta = \epsilon^{-1}$. Now the left-hand side divided by $\epsilon\delta^2$ is

$$\frac{\epsilon\delta^2 X^2 + \delta X - 1}{\epsilon\delta^2} = X^2 + X + \text{small}.$$

This can balance the zero on the right-hand side and gives the singular root $X = -1+$ small. (Note that the solution $X = 0$ is not permitted since X has to be strictly of order one).

(v) $\delta \gg \epsilon^{-1}$. Finally, if δ is larger still then the left-hand side divided by $\epsilon\delta^2$ is dominated by the first term

$$\frac{\epsilon\delta^2 X^2 + \delta X - 1}{\epsilon\delta^2} = X^2 + \text{small} + \text{small},$$

which cannot balance the zero on the right-hand side with X strictly of order one.

Alternative approach: pairwise comparison

An alternative method is to compare terms pairwise, which is quicker when there are a small number of terms. To get a sensible answer from equating the left-hand side to zero we need at least two terms to be in balance (sometimes known as a **distinguished limit**). The possible combinations are the first and second terms, the first and third terms, or the second and third terms.

(i) First and second terms in balance. To have ϵx^2 and x in balance requires x to be of size ϵ^{-1}. Then these terms are both of size ϵ^{-1}, and dominate the remaining term -1, which is of size one. This leads to the singular root.

(ii) First and third terms in balance. To have ϵx^2 and -1 in balance requires x to be of size $\epsilon^{-1/2}$. Then these terms are both of size one, but the remaining term x is of size $\epsilon^{-1/2}$, so that this single term dominates and there is no sensible balance.

(iii) Second and third terms in balance. To have x and -1 in balance requires x to be of size one. Then these terms are both of size one, and dominate the remaining term which is size ϵ. This leads to the regular root.

Non-integral powers

Consider the quadratic equation

$$(1-\epsilon)x^2 - 2x + 1 = 0.$$

Setting $\epsilon = 0$ gives $x = 1$ as the double root (a sign of the danger to come). Proceeding as usual we pose the expansion

$$x = 1 + \epsilon x_1 + \epsilon^2 x_2 + \cdots.$$

Substituting into the equation

$$(1-\epsilon)\left(1 + \epsilon x_1 + \epsilon^2 x_2 + \cdots\right)^2 - 2\left(1 + \epsilon x_1 + \epsilon^2 x_2 + \cdots\right) + 1 = 0.$$

Expanding gives

$$1 + 2x_1\epsilon + (2x_2 + x_1^2)\epsilon^2 + \cdots - \epsilon - 2x_1\epsilon^2 + \cdots - 2 - 2x_1\epsilon - 2x_2\epsilon^2 + \cdots + 1 = 0.$$

Comparing coefficients of ϵ gives

At ϵ^0: $\quad 1 - 2 + 1 = 0,$

which is automatically satisfied because we started with the correct value $x = 1$ at $\epsilon = 0$.

At ϵ^1: $\quad 2x_1 - 1 - 2x_1 = 0,$

which cannot be satisfied by any value of x_1 (except $x_1 = \infty$ in some sense).

The cause of the difficulty is illustrated by looking at the exact solution

$$x = \frac{1}{1 \pm \epsilon^{1/2}}.$$

Expanding the largest root for small ϵ gives

$$x = 1 + \epsilon^{1/2} + \epsilon + \epsilon^{3/2} + \cdots.$$

We should have expanded in powers of $\epsilon^{1/2}$ instead of ϵ. This is what $x_1 = \infty$ is hinting at: the scaling on x_1 is too small. (In retrospect we could have guessed that an order $\epsilon^{1/2}$ change in x would be required to produce and order ϵ change in a function at its minimum.)

If we pose the expansion

$$x = 1 + \epsilon^{1/2}x_{1/2} + \epsilon x_1 + \cdots,$$

and substitute into the equation

$$(1-\epsilon)\left(1 + \epsilon^{1/2}x_{1/2} + \epsilon x_1 + \cdots\right)^2 - 2\left(1 + \epsilon^{1/2}x_{1/2} + \epsilon x_1 + \cdots\right) + 1 = 0.$$

Expanding gives

$$1 + 2x_{1/2}\epsilon^{1/2} + (2x_1 + x_{1/2}^2)\epsilon + (2x_{3/2} + 2x_{1/2}x_1)\epsilon^{3/2} + \cdots - \epsilon - 2x_{1/2}\epsilon^{3/2} + \cdots$$
$$-2 - 2x_{1/2}\epsilon^{1/2} - 2x_1\epsilon - 2x_{3/2}\epsilon^{3/2} + \cdots + 1 = 0.$$

Comparing coefficients of ϵ we find that

At ϵ^0: $\quad 1 - 2 + 1 = 0,$

is automatically satisfied as usual and

At $\epsilon^{1/2}$: $\quad 2x_{1/2} - 2x_{1/2} = 0,$

is satisfied for all values of $x_{1/2}$. Slightly disturbing that $x_{1/2}$ is not determined but at least the expansion is consistent so far.

At ϵ^1: $\quad 2x_1 + x_{1/2}^2 - 1 - 2x_1 = 0,$

so that $x_{1/2} = \pm 1$ and x_1 is not determined at this level.

At $\epsilon^{3/2}$: $\quad 2x_{3/2} + 2x_{1/2}x_1 - 2x_{1/2} - 2x_{3/2} = 0,$

so that $x_1 = 1$ for both roots $x_{1/2}$, while $x_{3/2}$ is not determined.

Finding the right expansion sequence

How would we determine the expansion sequence if we did not have the exact solution to compare with? First pose a general expansion

$$x = 1 + \delta_1 x_1, \qquad \delta_1(\epsilon) \ll 1$$

and substitute this into the equation to get

$$(1-\epsilon)(1+\delta_1 x_1)^2 - 2(1+\delta_1 x_1) + 1 = 0.$$

Expanding

$$1 + 2\delta_1 x_1 + \delta_1^2 x_1^2 - \epsilon + 2\epsilon\delta_1 x_1 + \delta_1^2 \epsilon x_1^2 - 2 - 2\delta_1 x_1 + 1 = 0.$$

Simplifying leaves

$$\delta_1^2 x_1^2 - \epsilon + 2\epsilon\delta_1 x_1 + \delta_1^2 \epsilon x_1^2 = 0.$$

Now we play the dominant balance game again. Since $\epsilon\delta_1 \ll \epsilon$ the leading terms are $\delta_1^2 x_1^2$ and ϵ. Thus to get a sensible balance we need $\delta_1 = \epsilon^{1/2}$. With this value for δ_1 we equate coefficients of ϵ to get

$$x_1^2 - 1 = 0, \quad i.e.\ x_1 = \pm 1.$$

To proceed to higher order we play the game again. Choosing $x_1 = 1$ for example, we now have

$$x = 1 + \epsilon^{1/2} + \delta_2 x_2, \qquad \delta_2(\epsilon) \ll \epsilon^{1/2}.$$

Substituting into the equation

$$(1-\epsilon)\left(1 + \epsilon^{1/2} + \delta_2 x_2\right)^2 - 2\left(1 + \epsilon^{1/2} + \delta_2 x_2\right) + 1 = 0.$$

Expanding

$$\begin{aligned} &1 + 2\epsilon^{1/2} + \epsilon + 2\delta_2 x_2 + 2\epsilon^{1/2}\delta_2 x_2 + \delta_2^2 x_2^2 \\ &\quad - \epsilon - 2\epsilon^{3/2} - \epsilon^2 - 2\epsilon\delta_2 x_2 - 2\epsilon^{3/2}\delta_2 x_2 - \epsilon\delta_2^2 x_2^2 - 2 - 2\epsilon^{1/2} - 2\delta_2 x_2 + 1 = 0. \end{aligned}$$

Simplifying leaves

$$2\epsilon^{1/2}\delta_2 x_2 + \delta_2^2 x_2^2 - 2\epsilon^{3/2} - \epsilon^2 - 2\epsilon\delta_2 x_2 - 2\epsilon^{3/2}\delta_2 x_2 - \epsilon\delta_2^2 x_2^2 = 0.$$

Since $\delta_2 \ll \epsilon^{1/2}$ the dominant term involving δ_2 is $2\epsilon^{1/2}\delta_2 x_2$. This must balance with $-2\epsilon^{3/2}$, giving $\delta_2 = \epsilon$ and $x_2 = 1$.

Iterative method

This is often very useful in cases where the expansion sequence is not known. Writing the original quad as

$$(x-1)^2 = \epsilon x^2,$$

we are led to the iterative process

$$x_{n+1} = 1 \pm \epsilon^{1/2} x_n.$$

Starting with $x_0 = 1$ the positive root gives

$$x_1 = 1 + \epsilon^{1/2}$$

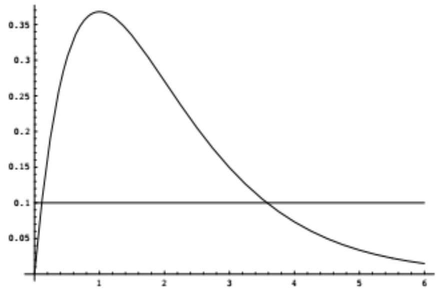
and

$$x_2 = 1 + \epsilon^{1/2} + \epsilon.$$

Logarithms

Consider the transcendental equation

$$xe^{-x} = \epsilon.$$



One root is near $x = 0$ and is easy to approximate. The other gets large as $\epsilon \to 0$ and is more difficult to find. Since the expansion sequence is not obvious we use the iterative procedure. Now, when $x = \log 1/\epsilon$, $xe^{-x} = \epsilon \log 1/\epsilon \gg \epsilon$. When $x = 2\log 1/\epsilon$, $xe^{-x} = 2\epsilon^2 \log 1/\epsilon \ll \epsilon$. Over this range the term x is slowly varying while e^{-x} is rapidly varying. This suggests rewriting the equation as

$$e^{-x} = \frac{\epsilon}{x}$$

giving the iterative scheme

$$x_{n+1} = \log(1/\epsilon) + \log x_n.$$

We have seen that the root lies roughly around $x = \log(1/\epsilon)$, so we start the iteration from $x_0 = \log(1/\epsilon)$. Then

$$x_1 = \log(1/\epsilon) + \log\log(1/\epsilon).$$

Then

$$\begin{aligned} x_2 &= \log(1/\epsilon) + \log(\log(1/\epsilon) + \log\log(1/\epsilon)) \\ &= \log(1/\epsilon) + \log\log(1/\epsilon) + \log\left(1 + \frac{\log\log(1/\epsilon)}{\log(1/\epsilon)}\right) \\ &= \log(1/\epsilon) + \log\log(1/\epsilon) + \frac{\log\log(1/\epsilon)}{\log(1/\epsilon)} - \frac{1}{2}\left(\frac{\log\log(1/\epsilon)}{\log(1/\epsilon)}\right)^2 + \cdots . \end{aligned}$$

Iterating again

$$
\begin{aligned}
x_3 &= \log(1/\epsilon) + \log\left(\log(1/\epsilon) + \log\log(1/\epsilon) + \frac{\log\log(1/\epsilon)}{\log(1/\epsilon)} - \frac{1}{2}\left(\frac{\log\log(1/\epsilon)}{\log(1/\epsilon)}\right)^2\right) \\
&= \log(1/\epsilon) + \log\log(1/\epsilon) + \log\left(1 + \frac{\log\log(1/\epsilon)}{\log(1/\epsilon)} + \frac{\log\log(1/\epsilon)}{(\log(1/\epsilon))^2} - \frac{1}{2}\frac{(\log\log(1/\epsilon))^2}{(\log(1/\epsilon))^3}\right) \\
&= \log(1/\epsilon) + \log\log(1/\epsilon) + \left(\frac{\log\log(1/\epsilon)}{\log(1/\epsilon)} + \frac{\log\log(1/\epsilon)}{(\log(1/\epsilon))^2} - \frac{1}{2}\frac{(\log\log(1/\epsilon))^2}{(\log(1/\epsilon))^3}\right) \\
&\quad - \frac{1}{2}\left(\frac{\log\log(1/\epsilon)}{\log(1/\epsilon)} + \frac{\log\log(1/\epsilon)}{(\log(1/\epsilon))^2} - \cdots\right)^2 + \frac{1}{3}\left(\frac{\log\log(1/\epsilon)}{\log(1/\epsilon)} + \cdots\right)^3 + \cdots \\
&= \log(1/\epsilon) + \log\log(1/\epsilon) + \frac{\log\log(1/\epsilon)}{\log(1/\epsilon)} + \\
&\quad \frac{-\frac{1}{2}(\log\log(1/\epsilon))^2 + \log\log(1/\epsilon)}{(\log(1/\epsilon))^2} + \frac{\frac{1}{3}(\log\log(1/\epsilon))^3 - \frac{3}{2}(\log\log(1/\epsilon))^2 + \cdots}{(\log(1/\epsilon))^3} + \cdots .
\end{aligned}
$$

Difficult sequence to guess. The appearance of $\log\epsilon$, and especially of $\log\log(1/\epsilon)$, means that very small values of ϵ are needed for the asymptotic expansion to be a good approximation. Normally we hope to do OK for $\epsilon = 0.5$, or at worst $\epsilon = 0.1$. However even when $\epsilon = 10^{-9}$, $\log\log(1/\epsilon)$ is only 3.