PERTURBATION METHODS

TUTORIAL TWO

Q1 Use Laplace's method to derive the leading-order asymptotic behaviour as $x \to \infty$ of the integrals

$$I_1(x) = \int_{-1}^{1} e^{-x\cosh t}\,\mathrm{d}t, \quad I_2(x) = \int_{-\pi/2}^{\pi/2} e^{-x(t^2-\sin^2 t)}\,\mathrm{d}t, \quad I_3(x) = \int_0^{\infty} e^{-2t-x/t^2}\,\mathrm{d}t.$$

[*You may assume that* $\int_0^\infty e^{-t^n}\,\mathrm{d}t = \Gamma(1/n)/n$ *for* $n = 2, 4$ and $\Gamma(1/2) = \sqrt{\pi}$.]

Q2 Use the method of stationary phase to derive the leading-order asymptotic behaviour as $x \to \infty$ of the integrals

$$J_1(x) = \int_0^1 \exp(ixt^2)\cosh(t^2)\,\mathrm{d}t, \quad J_2(x) = \int_0^1 \cos(xt^4)\tan(t)\,\mathrm{d}t, \quad J_3(x) = \int_0^1 \exp[ix(t-\sin t)]\,\mathrm{d}t.$$

[*You may assume that* $\int_0^\infty e^{it^n}\,\mathrm{d}t = e^{i\pi/2n}\Gamma(1/n)/n$ *for* $n = 2, 3$ and $\int_0^\infty te^{it^4}\,\mathrm{d}t = e^{i\pi/4}\Gamma(1/4)/4$.]

Q3 In this problem, you will use the method of steepest descents to derive the leading-order asymptotic behaviour as $x \to \infty$ of the integral

$$I(x) = \int_{-1}^{1} (1-t^2)^N e^{ixt}\,\mathrm{d}t,$$

where N is an integer and the contour of integration is a line segment from $t = -1$ to $t = 1$.

(a) Find and sketch in the complex t-plane the steepest descent contours through $t = \pm 1$.

(b) By deforming the contour of integration to a new contour that goes through both steepest descent contours, show that $I(x) = I_-(x) - I_+(x)$, where

$$I_\pm(x) = \int_{\pm 1}^{\pm 1 + i\infty} (1-t^2)^N e^{ixt}\,\mathrm{d}t.$$

(c) Use Laplace's method to derive the leading-order asymptotic behaviour as $x \to \infty$ of the integrals $I_\pm(x)$, and hence of $I(x)$.

[*You may assume that* $\Gamma(m+1) = \int_0^\infty t^m e^{-t}\,\mathrm{d}t = m!$ *for integer* m.]

Q4 Consider the error function

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2}\,\mathrm{d}s = \frac{2r}{\sqrt{\pi}}\int_0^{e^{i\theta}} e^{-r^2t^2}\,\mathrm{d}t,$$

where we have substituted $z = re^{i\theta}$ and $s = rt$. Use the method of steepest descents to derive the leading-order asymptotic behaviour of erf(z) as $r = |z| \to \infty$ for $0 < \theta < \pi/2$, distinguishing carefully between the cases $0 < \theta \le \pi/4$ and $\pi/4 < \theta < \pi/2$.

Q5 Let $I(\epsilon) = \int_0^1 f(x)/(x+\epsilon)\,dx$, where $\epsilon > 0$ and f is smooth. By writing $\int_0^1 = \int_0^\delta + \int_\delta^1$, where $\epsilon \ll \delta \ll 1$, show that

$$I(\epsilon) \sim -f(0)\log\epsilon + \int_0^1 \frac{f(x)-f(0)}{x}\,dx + \cdots \quad \text{as } \epsilon \to 0^+.$$

Q6 State which method or methods could be used to find the asymptotic behaviour of the following integrals in which x is real:

$$\int_0^{\pi/2} e^{ix\cos t}\mathrm{d}t, \quad \int_0^1 \ln t\, e^{ixt}\mathrm{d}t, \quad \int_0^x t^{-1/2}e^{-t}\mathrm{d}t, \quad \int_0^{\pi/2} e^{-x\sin^2 t}\mathrm{d}t, \quad \int_0^1 \exp\left(ixe^{-1/t}\right)\mathrm{d}t \quad \text{as } x \to \infty;$$

$$\int_0^{10} \frac{e^{-xt}}{1+t}\mathrm{d}t, \quad \int_0^{\pi/2} \frac{\mathrm{d}t}{\sqrt{\cos^2 t + x\sin^2 t}}, \quad \int_0^1 \frac{\sin(tx)}{t}\mathrm{d}t, \quad \int_x^\infty t^{a-1}e^{-t}\mathrm{d}t, \quad \int_0^1 \frac{\ln t}{x+t}\mathrm{d}t \quad \text{as } x \to 0^+.$$

You need not evaluate the asymptotic expansions.