PERTURBATION METHODS
FINAL EXAM

ATTEMPT ALL QUESTIONS (15 MARKS PER QUESTION)

Q1 (a) Show that $\ddot{x}+\epsilon\dot{x}+x=0$ has a multiple scales solution of the form

$$x \sim \frac{1}{2}\left(A(T)e^{it}+\overline{A}(T)e^{-it}\right) \quad \text{as } \epsilon \to 0^+ \text{ with } T=\epsilon t = \mathrm{O}(1), \tag{1}$$

where A is a complex function of T that you should determine and $\overline{A}$ denotes the complex conjugate of A. By writing $A(T)=R(T)e^{i\Theta(T)}$, where $R \geq 0$, show that the result agrees with the expansion of the exact solution for $t=O(1/\epsilon)$.

(b) Show that $\ddot{x}+x=\epsilon x^3$ has a multiple scales solution of the form (1) provided $A(T)$ satisfies a differential equation that you should determine. Hence, determine $A(T)$.

(c) Show that the van der Pol equation $\ddot{x}+\epsilon(x^2-\lambda)\dot{x}+x=0$ has a multiple scales solution of the form (1) provided $A(T)$ satisfies a differential equation that you should determine. Show that as λ increases through zero a periodic solution is born in which x is approximately sinusoidal in t, with period 2π and amplitude $2\sqrt{\lambda}$.

Q2 (a) Show that $\ddot{x}+(1+\epsilon)x=\cos t$ has a multiple scales solution of the form

$$x \sim \frac{1}{2\epsilon}\left(A(T)e^{it}+\overline{A}(T)e^{-it}\right) \quad \text{as } \epsilon \to 0^+ \text{ with } T=\epsilon t = \mathrm{O}(1) \tag{2}$$

provided $A(T)$ satisfies a differential equation that you should determine. When is the leading-order multiple scales solution periodic with period 2π?

(b) Show that the Duffing equation $\ddot{x}+(1+\epsilon)x+\kappa\epsilon^3x^3=\cos t$, where κ is a real positive constant, has a multiple scales solution of the form (2) provided $A(T)$ satisfies a differential equation that you should determine. When is the leading-order multiple scales solution periodic with period 2π?

Q3 Consider the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(D\left(x,\frac{x}{\epsilon}\right)\frac{\mathrm{d}u}{\mathrm{d}x}\right)=f\left(x,\frac{x}{\epsilon}\right).$$

where $D(x,X)>0$ and $f(x,X)$ are smooth and periodic in X with period one. Determine the PDEs satisfied by u_0, u_1 and u_2 in the multiple scales expansion $u \sim u_0(x,X)+\epsilon u_1(x,X)+\epsilon^2u_2(x,X)+\cdots$ as $\epsilon \to 0^+$ with $X=x/\epsilon=\mathrm{O}(1)$. Deduce that, if u_0, u_1 and u_2 are periodic in X with period one, then u_0 is a function only of x satisfying a second-order ODE that you should determine.

Q4 Determine the leading-order term in the WKB expansions $y(x) \sim A(x)e^{iu(x)/\epsilon}$ as $\epsilon \to 0^+$ for the two independent solutions of (a) $\epsilon^2y''+xy=0$ for $x>0$; (b) $\epsilon^2y''-xy=0$ for $x>0$. How close to $x=0$ do you have to be for these expansions to lose their validity?

Q5 The function $y(x)$ satisfies $\epsilon y''+y'+xy=0$ for $0<x<1$, with $y(0)=0$ and $y(1)=1$, where $\epsilon>0$.

(a) Obtain a two-term approximation using a WKB expansion of the form $y=e^{S(x)/\epsilon}$, with $S(x) \sim S_0(x)+\epsilon S_1(x)+\cdots$ as $\epsilon \to 0^+$.

(b) Use boundary layer theory to analyse the problem as $\epsilon \to 0^+$. Determine the positions and scalings of the boundary layer(s) and find the leading-order outer and inner solutions. Match the outer and inner solutions. Hence determine a leading-order additive composite expansion.

Q6 The function $y(x)$ satisfies $\epsilon^2y''+(1-x)y=0$ for $x>0$, with $y(0)=1$ and $y(\infty)=0$, where $\epsilon>0$.

(a) By making the change of variable $x=1+\epsilon^{2/3}X$, find the exact solution $y(x)$ using Airy functions.

(b) Use WKB theory and the method of matched asymptotic expansions to find the leading-order asymptotic solution for $x-1=O(1)$ and $X=O(1)$ as $\epsilon \to 0^+$.

[*You may quote the asymptotic behaviour of the Airy functions* $Ai(X)$ *and* $Bi(X)$ *as* $X \to \pm\infty$.]