PERTURBATION METHODS

CAT (5 MARKS PER QUESTION): ATTEMPT ALL QUESTIONS

Q1 (a) Write out in words Van Dyke's matching rule "(m.t.i.)(n.t.o.) = (n.t.o.)(m.t.i.)".

(b) Find and match for $(m, n) = (1,1)$, $(1,2)$, $(2,1)$ and $(2,2)$ the expansions of the function $\sqrt{1+\sqrt{x+\epsilon}}$ as $\epsilon \to 0^+$ with $x = O(1)$ and $X = x/\epsilon = O(1)$.

(c) Find expansions of the function $1+\log x/\log \epsilon$ as $\epsilon \to 0^+$ with $x = O(1)$ and $X = x/\epsilon = O(1)$. Check that matching for $m = n = 1$ does not work and suggest how to resolve this situation.

Q2 For each of the following problems find and match two terms of the outer and inner expansions, $y \sim y_0(x) + \epsilon y_1(x) + \cdots$ and $y \sim Y_0(X) + \epsilon Y_1(X) + \cdots$, respectively, where $X = x/\epsilon = \mathrm{O}(1)$ as $\epsilon \to 0^+$. In particular, show that in case (a) the matching is automatic in the sense it does not determine any of the constants of integration and that in case (b) $y_0 = 0$.

(a) $\epsilon y' + y = x$ for $x > 0$, with $y(0) = 1$;

(b) $(x+\epsilon)y' + y = 0$ for $x > 0$, with $y(0) = 1$.

Q3 Consider as $\epsilon \to 0^+$ the problem $\epsilon y'' + x^{1/2}y' + y = 0$ for $0 < x < 1$, with $y(0) = 0$ and $y(1) = 1$.

(a) Show that there can be no boundary layer at $x = 1$.

(b) Show that in the outer region $y \sim e^{2(1-x^{1/2})}$ for $x = \mathrm{O}(1)$ as $\epsilon \to 0^+$.

(c) Show that there is a boundary layer of thickness of $\mathrm{O}(\epsilon^{2/3})$ at $x = 0$ in which the first two terms of the differential equation are in balance.

(d) Match to show that in the inner region $y \sim C \int_0^X e^{-2t^{3/2}/3}\,\mathrm{d}t$, where $X = \epsilon^{-2/3}x = \mathrm{O}(1)$ as $\epsilon \to 0^+$ and C is a constant that you should determine in terms of the gamma function.

Q4 (a) Consider as $\epsilon \to 0^+$ the problem $\epsilon y'' + yy' - y = 0$ for $0 < x < 1$, with $y(0) = 1$ and $y(1) = 3$. Assuming that there is a boundary layer only near $x = 0$, find the leading-order terms in the outer and inner expansions and match them.

(b) Consider as $\epsilon \to 0^+$ the problem $\epsilon y'' + yy' - y = 0$ for $0 < x < 1$, with $y(0) = -3/4$ and $y(1) = 5/4$, in which the boundary layer is at an interior position. Find and match the leading order terms in the outer and inner expansions and determine the position of the interior layer.

Q5 Consider as $\epsilon \to 0^+$ the problem $y'' + \epsilon y' = 0$ for $0 < x < L$, with $y(0) = 0$ and $y(L) = 1$.

(a) If $L = \mathrm{O}(1)$ as $\epsilon \to 0^+$, show that

$$y \sim \frac{x}{L} + \epsilon \frac{x(L-x)}{2L} + \cdots \qquad \text{as } \epsilon \to 0^+.$$

(b) For large values of L this expansion gives $y'(0) = \epsilon/2$, but the exact solution is $y = (1-e^{-\epsilon x})/(1-e^{-\epsilon L})$, giving $y'(0) = \epsilon$ as $L \to \infty$. Explain.

Q6 (a) Suppose $\epsilon \nabla^2 u = u$ in $r^2 = x^2 + y^2 < 1$ with $u = 1$ on $r = 1$. Show that a formal boundary layer analysis as $\epsilon \to 0^+$ gives $u = e^{-R} + O(\epsilon^{1/2})$ for $R = \epsilon^{-1/2}(1-r) = O(1)$ and $u = o(\epsilon^n)$ for all $n \in \mathbb{N}$ for $1-r = O(1)$. Verify the formal result by expanding the exact solution, which you may assume to be given by $u = I_0(r/\sqrt{\epsilon})/I_0(1/\sqrt{\epsilon})$, where I_0 is the modified Bessel function

$$I_0(x) = \frac{1}{\pi}\int_0^\pi \cos(ix\sin\theta)\,d\theta.$$

(b) Suppose $\epsilon \nabla^2 u = u_x$ in $y > 0$, with $u = 1$ on $y = 0$, $x > 0$; $u_y = 0$ on $y = 0$, $x < 0$; and $u \to 0$ as $x^2 + y^2 \to \infty$, $y > 0$. Show that a formal boundary layer analysis as $\epsilon \to 0^+$ gives

$$u = \operatorname{erfc}\left(\frac{Y}{2\sqrt{x}}\right) + O(\epsilon) \text{ for } Y = \frac{y}{\sqrt{\epsilon}} = O(1), \quad x > 0$$

and $u = o(\epsilon^n)$ for all $n \in \mathbb{N}$ almost everywhere else. Where does u satisfy neither of these approximations?