Steady two-dimensional heat flow is governed by Laplace's equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0. \tag{5.1}$$

If r and θ are the usual plane polar coordinates,

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{5.2}$$

Laplace's equation becomes

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0. \tag{5.3}$$

For rectangles, discs, the exteriors of discs, and annuli, we can use separation of variables and Fourier series to construct solutions of (5.1) and (5.3).

BVP in cartesian coordinates

Consider the following BVP: find the solution of Laplace's equation (5.1) which is defined on the rectangle $\{(x,y) : 0 \le x \le a, 0 \le y \le b\}$ and satisfies the boundary conditions

$$T(0,y) = T(a,y) = 0, \quad 0 < y < b, \tag{5.4}$$

$$T(x,0) = 0 \text{ and } T(x,b) = f(x), \quad 0 < x < a. \tag{5.5}$$

To begin we look for special solutions of Laplace's equation of the separable form $T(x,y) = F(x)G(y)$. Substituting into (5.1) and dividing through by $F(x)G(y)$ gives

$$\frac{1}{F(x)}F''(x) + \frac{1}{G(y)}G''(y) = 0. \tag{5.6}$$

Hence there is a constant λ such that

$$F''(x) = -\lambda F(x), \quad G''(y) = \lambda G(y). \tag{5.7}$$

The choices $\lambda = n^2\pi^2/a^2$, $F(x) = \sin(n\pi x/a)$ $(n = 1, 2, 3...)$ give the solutions

$$T(x,y) = \sin\left(\frac{n\pi x}{a}\right) G(y), \tag{5.8}$$

which satisfy the boundary conditions (5.3). Furthermore $G(y)$ is a solution of the ODE

$$G''(y) = \frac{n^2 y^2}{a^2} G(y), \tag{5.9}$$

and so

$$G(y) = A\cosh\left(\frac{n\pi y}{a}\right) + B\sinh\left(\frac{n\pi y}{a}\right), \tag{5.10}$$

in which hyperbolic functions, rather than trigonometric functions, occur. The choice $A = 0$ ensures that the boundary condition $T(x, 0) = 0$ $(0 < x < a)$ holds and thus we are led to consider solutions of the BVP of the form

$$T(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right). \tag{5.11}$$

On setting $y = b$ we see that the coefficients B_n are determined by the condition that

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi b}{a}\right) = f(x), \quad 0 < x < a. \tag{5.12}$$

Hence

$$B_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a f(s)\sin\left(\frac{n\pi s}{a}\right) \mathrm{d}s, \tag{5.13}$$

and the BVP has the solution

$$T(x,y) = \frac{2}{a}\sum_{n=1}^{\infty} \frac{1}{\sinh(n\pi b/a)} \left[\int_0^a f(s)\sin\left(\frac{n\pi s}{a}\right)\mathrm{d}s\right] \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right). \tag{5.14}$$

BVP in polar coordinates

Next, let us consider separable solutions of Laplace's equation of the form $T(r,\theta) = F(r)G(\theta)$. Substitution into (5.3) gives

$$F''(r)G(\theta) + \frac{1}{r}F'(r)G(\theta) + \frac{1}{r^2}F(r)G''(\theta) = 0. \tag{5.15}$$

Hence

$$\frac{r^2F''(r) + rF'(r)}{F(r)} + \frac{G''(\theta)}{G(\theta)} = 0, \tag{5.16}$$

and there is a constant λ such that

$$r^2F''(r) + rF'(r) = \lambda F(r), \tag{5.17}$$

$$G''(\theta) = -\lambda G(\theta). \tag{5.18}$$

The function $G(\theta)$ must be periodic with period 2π so that $G(\theta + 2\pi) = G(\theta)$, and this is possible only if $\lambda = n^2$, where n is an integer. If $n = 0$, the only solutions of (5.18) which are periodic are $G(\theta) \equiv$ constant. If $n \neq 0$ the periodic solutions are arbitrary linear combinations of $\cos(n\theta)$ and $\sin(n\theta)$. When $n=0$ the solutions of (5.17) are of the form $F(r) = A + B\log r$. When $n \neq 0$, equation (5.17) is of Euler's type and $F(r)$ is a linear combination of r^n and r^{-n}. Thus there are separable solutions of Laplace's equation of the forms

$$T(r) = A + B\log r, \tag{5.19}$$

and

$$T(r,\theta) = \left(Ar^n + Br^{-n}\right)\left[C\cos(n\theta) + D\sin(n\theta)\right], \tag{5.20}$$

where n is a positive integer and A, B, C, D are arbitrary constants. The solutions $\log r$, $r^{-n}\cos n\theta$, $r^{-n}\sin n\theta$ are not defined at $r=0$ and so are not admissible if the origin belongs to the domain in which T is defined.

Example 5.1 Find T so as to satisfy Laplace's equation in the annulus $a < r < b$ and the boundary conditions

$$T = T_0 \text{ on } r = a, \quad T = T_1 \text{ on } r = b, \tag{5.21}$$

where T_0 and T_1 are constants. By inspection, $T = A + B\log r$, where A and B must satisfy

$$A + B\log a = T_0, \quad A + B\log b = T_1. \tag{5.22}$$

Then

$$A = \frac{T_0\log b - T_1\log a}{\log(b/a)}, \quad B = \frac{T_1 - T_0}{\log(b/a)}, \tag{5.23}$$

and

$$T(r,\theta) = \frac{T_0\log b - T_1\log a}{\log(b/a)} + \frac{T_1 - T_0}{\log(b/a)}\log r. \tag{5.24}$$

Example 5.2 A conductor occupies the region $r \leq a$ and the temperature satisfies the boundary condition $T(a,\theta) = \sin^3\theta$. Find $T(r,\theta)$ in $r < a$.

Note that

$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin(3\theta). \tag{5.25}$$

Hence

$$T = \frac{3}{4}\left(\frac{r}{a}\right)\sin\theta - \frac{1}{4}\left(\frac{r}{a}\right)^3\sin(3\theta). \tag{5.26}$$

Application of Fourier series

We wish to find T so as to satisfy Laplace's equation in the disc $0 \leq r < a$ and the boundary condition $T = f(\theta)$ on $r = a$, $(0 \leq \theta \leq 2\pi)$, where f is a prescribed function. Here the solution is of the form

$$T(r,\theta) = A + \sum_{n=1}^{\infty} r^n \left[C_n \cos(n\theta) + D_n \sin(n\theta)\right], \tag{5.27}$$

and the boundary condition gives

$$A + \sum_{n=1}^{\infty} a^n \left[C_n \cos(n\theta) + D_n \sin(n\theta)\right] = f(\theta), \quad 0 \leq \theta \leq 2\pi. \tag{5.28}$$

Thus

$$
\begin{aligned}
A &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta, && (5.29)\\
C_n &= \frac{1}{\pi a^n}\int_0^{2\pi} f(\theta)\cos(n\theta)\,\mathrm{d}\theta, && (5.30)\\
D_n &= \frac{1}{\pi a^n}\int_0^{2\pi} f(\theta)\sin(n\theta)\,\mathrm{d}\theta. && (5.31)
\end{aligned}
$$

Example 5.3 Find $T(r,\theta)$ so as to satisfy Laplace's equation in the disc $r < a$ and the boundary condition

$$T(a,\theta) = |\sin\theta|, \quad 0 \le \theta \le 2\pi. \tag{5.32}$$

The solution is

$$T(r,\theta) = A + \sum_{n=1}^{\infty} r^n \left[C_n\cos(n\theta) + D_n\sin(n\theta)\right], \tag{5.33}$$

where

$$
\begin{aligned}
A &= \frac{1}{2\pi}\int_0^{2\pi}|\sin\theta|\,\mathrm{d}\theta = \frac{1}{2\pi}\left[\int_0^{\pi}\sin\theta\,\mathrm{d}\theta - \int_{\pi}^{2\pi}\sin\theta\,\mathrm{d}\theta\right] = \frac{2}{\pi}, && (5.34)\\
C_n &= \frac{1}{\pi a^n}\int_0^{2\pi}|\sin\theta|\cos(n\theta)\,\mathrm{d}\theta = \begin{cases} 0 & n \text{ odd},\\ -4/(\pi a^n(n^2-1)) & n \text{ even}, \end{cases} && (5.35)\\
D_n &= \frac{1}{\pi a^n}\int_0^{2\pi}|\sin\theta|\sin(n\theta)\,\mathrm{d}\theta = 0, && (5.36)
\end{aligned}
$$

and so

$$T(r,\theta) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^{2n}\frac{\cos(2n\theta)}{4n^2-1}. \tag{5.37}$$

5.2.2 Poisson's formula

Consider again the problem from Section 5.2.1: find T so as to satisfy Laplace's equation in the disc $0 \le r < a$ and the boundary condition $T = f(\theta)$ on $r = a$, $(0 \le \theta \le 2\pi)$, where f is a prescribed function. Poisson's formula states that the solution to this problem can be written

$$T(r,\theta) = \frac{(a^2-r^2)}{2\pi}\int_0^{2\pi}\frac{f(\phi)}{a^2+r^2-2ar\cos(\theta-\phi)}\,\mathrm{d}\phi. \tag{5.38}$$

Lemma 5.1 If λ and α are real and $|\lambda| < 1$ then

$$\frac{1}{2} + \sum_{n=1}^{\infty}\lambda^n\cos n\alpha = \frac{1-\lambda^2}{2(1+\lambda^2-2\lambda\cos\alpha)}. \tag{5.39}$$

Proof.

$$\sum_{n=1}^{\infty} \lambda^n \cos n\alpha = R_e \sum_{n=1}^{\infty} \lambda^n e^{in\alpha}, \tag{5.40}$$

$$= R_e \sum_{n=1}^{\infty} (\lambda e^{i\alpha})^n, \tag{5.41}$$

$$= R_e \left[\frac{\lambda e^{i\alpha}}{1-\lambda e^{i\alpha}}\right], \tag{5.42}$$

$$= \frac{\lambda \cos\alpha - \lambda^2}{1+\lambda^2 - 2\lambda\cos\alpha}. \tag{5.43}$$

Hence

$$\frac{1}{2} + \sum_{n=1}^{\infty} \lambda^n \cos n\alpha = \frac{1-\lambda^2}{2(1+\lambda^2-2\lambda\cos\alpha)}. \tag{5.44}$$

□

Proof of Poisson's formula. Taking care over the dummy variable of integration we get

$$T(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,\mathrm{d}\phi \tag{5.45}$$

$$+\frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \left[\cos(n\theta)\int_0^{2\pi} f(\phi)\cos(n\phi)\,\mathrm{d}\phi + \sin(n\theta)\int_0^{2\pi} f(\phi)\sin(n\phi)\,\mathrm{d}\phi\right], \tag{5.46}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos[n(\theta-\phi)]\right\} f(\phi)\,\mathrm{d}\phi, \tag{5.47}$$

and, on appealing to Lemma 5.1, with $\lambda = r/a$ and $\alpha = \theta - \phi$, the result follows. □

Corollary 5.2 (Mean value property of solution of Laplace's equation) The value of T at the centre of the disc is

$$T(0,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,\mathrm{d}\phi = \text{mean value over boundary}. \tag{5.48}$$

Uniqueness

Uniqueness is established with the aid of *Green's Theorem*:

Theorem 5.3 (Green's theorem) If R is a bounded and connected plane region whose boundary ∂R is the union $C_1 \cup \ldots \cup C_n$ of a finite number of simple closed curves, oriented so that R is on the left, and if $p(x,y)$ and $q(x,y)$ are continuous and have continuous first derivatives on $R \cup \partial R$, then

$$\int_{\partial R} p\,\mathrm{d}x + q\,\mathrm{d}y = \int\int_R \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y. \tag{5.49}$$

In the figure, R is the shadowed region. It has two 'holes' and ∂R is the union of three simple closed curves oriented as shown.

Uniqueness for the Dirichlet problem

We consider uniqueness of solutions to the Dirichlet problem, working in Cartesian coordinates.

Consider the BVP

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \text{ in } R, \quad T = f \text{ on } \partial R, \tag{5.50}$$

where f is a prescribed function and R is a bounded and connected region as in the statement of Green's theorem. Then the BVP has at most one solution.

Proof. Let S also be a solution, so that

$$\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} = 0 \text{ in } R, \quad S = f \text{ on } \partial R. \tag{5.51}$$

Then the difference $W := T - S$ in a solution of the BVP

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} = 0 \text{ in } R, \quad W = 0 \text{ on } \partial R. \tag{5.52}$$

Consider the identity

$$W\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}\right) + \left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 = \frac{\partial}{\partial x}\left(W\frac{\partial W}{\partial x}\right) + \frac{\partial}{\partial y}\left(W\frac{\partial W}{\partial y}\right). \tag{5.53}$$

Integrate both sides over R and appeal to Laplace's equation and Green's theorem to find that

$$\begin{aligned}\int\int_R \left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2\right] \mathrm{d}x\,\mathrm{d}y &= \int\int_R \frac{\partial}{\partial x}\left[\left(W\frac{\partial W}{\partial x}\right) + \frac{\partial}{\partial y}\left(W\frac{\partial W}{\partial y}\right)\right] \mathrm{d}x\,\mathrm{d}y, \\ &= \int_{\partial R}\left[-W\frac{\partial W}{\partial y}\,\mathrm{d}x + W\frac{\partial W}{\partial x}\,\mathrm{d}y\right]. \end{aligned} \tag{5.54}$$

Since $W = 0$ on ∂R the line integral must vanish and so

$$\int\int_R \left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2\right] \mathrm{d}x\,\mathrm{d}y = 0. \tag{5.55}$$

This is possible only if $\partial W/\partial x = 0$, $\partial W/\partial y = 0$ in R. Hence W is constant and since $W = 0$ on ∂R the constant can only equal zero. Hence $T = S$ and the solution is unique. □

Example 5.4 We now consider a BVP in an unbounded region, for which our uniqueness proof does not apply. A conductor occupies the region $r \geq a$, *i.e.* that exterior to the circle $r = a$, and T satisfies the boundary condition $T = x$ on $r = a$. Find T in $r > a$, given that T remains bounded as $r \to \infty$.

In terms of polar coordinates the boundary condition is $T = a\cos\theta$ for $r = a$ and $0 \leq \theta \leq 2\pi$. This suggests that we seek a solution of the form

$$T = \left(Ar + \frac{B}{r}\right)\cos\theta. \tag{5.56}$$

If T is to remain bounded as $r \to \infty$ we must have $A = 0$, and to match the boundary condition on $r = a$ we must have $B = a^2$. Hence the solution is

$$T(r,\theta) = \frac{a^2\cos\theta}{r}, \tag{5.57}$$

or, in Cartesian coordinates,

$$T(x,y) = \frac{a^2 x}{x^2 + y^2}. \tag{5.58}$$

Example 5.5 (Poisson's equation) The same argument establishes uniqueness for the BVP:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = F(x,y) \text{ in } R, \quad T = f(x,y) \text{ on } \partial R, \tag{5.59}$$

where F and f are prescribed functions.

Uniqueness for the Neumann problem

Firstly, we consider the Neumann problem in Cartesian coordinates.

Theorem 5.5 Consider the BVP

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = F(x,y) \text{ in } R, \quad \frac{\partial T}{\partial n} = g(x,y) \text{ on } \partial R, \tag{5.60}$$

where F and g are prescribed functions. Then the BVP has no solution unless

$$\int\int_R F \,\mathrm{d}x\,\mathrm{d}y = \int_{\partial R} g \,\mathrm{d}s. \tag{5.61}$$

When a solution exists it is not unique; all solutions differ by a constant.

Proof. For the first part we use Green's theorem

$$\int\int_R \underbrace{\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)}_{F} \mathrm{d}x\,\mathrm{d}y = \int_{\partial R} \frac{\partial T}{\partial x}\mathrm{d}y - \frac{\partial T}{\partial y}\mathrm{d}x = \int_{\partial R} \underbrace{\frac{\partial T}{\partial n}}_{g} \mathrm{d}s. \tag{5.62}$$

For the second part, let U be a solution of the same BVP *i.e.*

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = F(x,y) \text{ in } R, \quad \frac{\partial U}{\partial n} = g(x,y) \text{ on } \partial R, \tag{5.63}$$

and consider $W := U - T$. Then W is a solution of the problem

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} = 0 \text{ in } R, \quad \frac{\partial W}{\partial n} = 0 \text{ on } \partial R. \tag{5.64}$$

Using the identity

$$W\left(\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}\right)+\left(\frac{\partial W}{\partial x}\right)^2+\left(\frac{\partial W}{\partial y}\right)^2=\frac{\partial}{\partial x}\left(W\frac{\partial W}{\partial x}\right)+\frac{\partial}{\partial y}\left(W\frac{\partial W}{\partial y}\right), \quad (5.65)$$

we find that

$$\begin{aligned}\int\int_R\left[\left(\frac{\partial W}{\partial x}\right)^2+\left(\frac{\partial W}{\partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y &= \int_{\partial R}-W\frac{\partial W}{\partial y}\,\mathrm{d}x+W\frac{\partial W}{\partial x}\,\mathrm{d}y, &(5.66)\\ &= \int_{\partial R}W\frac{\partial W}{\partial n}\,\mathrm{d}s, &(5.67)\\ &= 0. &(5.68)\end{aligned}$$

Thus $\partial W/\partial x=0$, $\partial W/\partial y=0$ so that $W=U-T$ is constant. □

Example 5.6 (Polar Neumann problem) Find T so as to satisfy Laplace's equation in the disk $0\le r<a$ and the boundary condition

$$\frac{\partial T}{\partial n}(a,\theta)=g(\theta), \quad 0\le\theta\le 2\pi, \quad (5.69)$$

where g is a prescribed function.

First, we define $\boldsymbol{n}=\cos\theta\boldsymbol{i}+\sin\theta\boldsymbol{j}$ and note that since $x=r\cos\theta$, $y=r\sin\theta$,

$$\begin{aligned}\frac{\partial T}{\partial r} &= \frac{\partial T}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial T}{\partial y}\frac{\partial y}{\partial r}, &(5.70)\\ &= \cos\theta\frac{\partial T}{\partial x}+\sin\theta\frac{\partial T}{\partial y}, &(5.71)\\ &= (\cos\theta\boldsymbol{i}+\sin\theta\boldsymbol{j})\cdot\left(\frac{\partial T}{\partial x}\boldsymbol{i}+\frac{\partial T}{\partial y}\boldsymbol{j}\right), &(5.72)\\ &= \boldsymbol{n}\cdot\nabla T, &(5.73)\\ &= \frac{\partial T}{\partial n}. &(5.74)\end{aligned}$$

[In this case $\partial T/\partial n$ is a genuine derivative.]

Let $g(\theta)$ have the Fourier expansion

$$g(\theta)=\frac{1}{2}p_0+\sum_{n=1}^{\infty}\left[p_n\cos(n\theta)+q_n\sin(n\theta)\right], \quad (5.75)$$

where

$$\begin{aligned}p_0 &= \frac{1}{\pi}\int_0^{2\pi}g(\theta)\,\mathrm{d}\theta, &(5.76)\\ p_n &= \frac{1}{\pi}\int_0^{2\pi}g(\theta)\cos(n\theta)\,\mathrm{d}\theta, &(5.77)\\ q_n &= \frac{1}{\pi}\int_0^{2\pi}g(\theta)\sin(n\theta)\,\mathrm{d}\theta. &(5.78)\end{aligned}$$

Look for solutions in the form

$$T(r,\theta)=A+\sum_{n=1}^{\infty}r^n\left[C_n\cos(n\theta)+D_n\sin(n\theta)\right]. \quad (5.79)$$

We have

$$\frac{\partial T}{\partial r} = \sum_{n=1}^{\infty} n r^{n-1} \left[C_n \cos(n\theta) + D_n \sin(n\theta) \right], \tag{5.80}$$

and the boundary condition gives

$$\sum_{n=1}^{\infty} n a^{n-1} \left[C_n \cos(n\theta) + D_n \sin(n\theta) \right] = g(\theta), \quad 0 \le \theta \le 2\pi. \tag{5.81}$$

We conclude immediately that the condition

$$\int_0^{2\pi} g(\theta)\, \mathrm{d}\theta = 0, \tag{5.82}$$

is necessary for a solution to exist.

If this condition is satisfied then there are solutions

$$T(r,\theta) = A + \sum_{n=1}^{\infty} r^n \left[C_n \cos(n\theta) + D_n \sin(n\theta) \right], \tag{5.83}$$

where

$$C_n = \frac{1}{n\pi a^{n-1}} \int_0^{2\pi} g(\theta) \cos(n\theta)\, \mathrm{d}\theta, \tag{5.84}$$

$$D_n = \frac{1}{n\pi a^{n-1}} \int_0^{2\pi} g(\theta) \sin(n\theta)\, \mathrm{d}\theta, \tag{5.85}$$

and A is an arbitrary constant, *i.e.* solutions are *non-unique*, if they exist.

Example 5.7 Find T so as to satisfy Laplace's equation in the disc $0 \le r < a$ and the boundary condition

$$\frac{\partial T}{\partial n}(a,\theta) = \sin^3\theta, \quad 0 \le \theta \le 2\pi. \tag{5.86}$$

Here

$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin(3\theta), \tag{5.87}$$

and

$$\int_0^{2\pi} \sin^3\theta\, \mathrm{d}\theta = 0, \tag{5.88}$$

and the solutions are

$$T(r,\theta) = A + \frac{3}{4} r \sin\theta - \frac{1}{12}\frac{r^3}{a^2}\sin(3\theta), \tag{5.89}$$

where A is arbitrary.

Well-posedness

PDE problems often arise from modelling a particular physical system. In this case we could like to be able to make predictions as to the behaviour of the system based on our analysis of the PDE under consideration.

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Definition A problem is said to be well-posed (well-set) if the following three conditions are satisfied:

1. EXISTENCE—there is a solution;
2. UNIQUENESS—there is no more than one solution;
3. CONTINUOUS DEPENDENCE—the solution depends continuously on the data.

Example 5.8 As an illustration consider the IVP

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad -\infty < x < \infty,\ t > 0, \tag{5.90}$$

$$y(x,0) = f(x), \quad \frac{\partial y}{\partial t}(x,0) = g(x), \quad -\infty < x < \infty, \tag{5.91}$$

where f and g are the initial data. We know that there is exactly one solution, given by D'Alembert's formula:

$$y(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s. \tag{5.92}$$

Thus 1. and 2. hold.

Suppose we are interested in making predictions in the time interval $0 < t < T$ for some time T. Consider a similar problem

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad -\infty < x < \infty,\ t > 0, \tag{5.93}$$

$$y(x,0) = F(x), \quad \frac{\partial y}{\partial t}(x,0) = G(x), \quad -\infty < x < \infty, \tag{5.94}$$

where F and G are different initial data. Again, we know that there is exactly one solution:

$$Y(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,\mathrm{d}s, \tag{5.95}$$

and

$$Y(x,t) - y(x,t) = \frac{1}{2}\left[(F(x-ct) - f(x-ct)) + (F(x+ct) - f(x+ct))\right] \tag{5.96}$$

$$+\frac{1}{2c}\int_{x-ct}^{x+ct} \left[G(s) - g(s)\right]\,\mathrm{d}s. \tag{5.97}$$

Now let $\epsilon > 0$ be arbitrary and suppose that

$$|F(x) - f(x)| < \delta \text{ and } |G(x) - g(x)| < \delta \text{ for } -\infty < x < \infty. \tag{5.98}$$

Then

$$\begin{aligned}
|Y(x,t)-y(x,t)| &\le \frac{1}{2}|F(x-ct)-f(x-ct)| \\
&\quad +\frac{1}{2}|F(x+ct)-f(x+ct)| \\
&\quad +\frac{1}{2c}\int_{x-ct}^{x+ct}|G(s)-g(s)|\,\mathrm{d}s, && (5.99)\\
&< \frac{1}{2}\delta+\frac{1}{2}\delta+\frac{1}{2c}\int_{x-ct}^{x+ct}\delta\,\mathrm{d}s, && (5.100)\\
&= \frac{1}{2}\delta+\frac{1}{2}\delta+\frac{1}{2c}\cdot 2ct\delta, && (5.101)\\
&= (1+t)\delta && (5.102)\\
&< (1+T)\delta. && (5.103)
\end{aligned}$$

Thus if the new data (F,G) are close to the original data (f,g) in the sense that

$$|F(x)-f(x)|<\frac{\epsilon}{1+T} \text{ and } |G(x)-g(x)|<\frac{\epsilon}{1+T} \quad \text{for } -\infty<x<\infty, \tag{5.104}$$

then the corresponding solutions are close together in the sense that

$$|Y(x,t)-y(x,t)|<\epsilon \quad \text{for } -\infty<x<\infty \text{ and } 0<t<T. \tag{5.105}$$

In this sense 3. holds and the IVP is well-posed.

Example 5.9 By contrast the corresponding IVP for Laplace's equation is *not* well-posed. If $y(x,t)=0,\ f(x)=0,\ g(x)=0$ then y is a solution of the IVP

$$\frac{\partial^2 y}{\partial x^2}+\frac{\partial^2 y}{\partial t^2}=0, \quad -\infty<x<\infty,\ t>0, \tag{5.106}$$

$$y(x,0)=f(x), \quad \frac{\partial y}{\partial t}(x,0)=g(x), \quad -\infty<x<\infty. \tag{5.107}$$

If

$$Y(x,t)=\delta^2\cos\left(\frac{x}{\delta}\right)\sinh\left(\frac{t}{\delta}\right),\ F(x)=0,\ G(x)=\delta\cos\left(\frac{x}{\delta}\right), \tag{5.108}$$

Then $Y(x,t)$ is a solution of the IVP

$$\frac{\partial^2 Y}{\partial x^2}+\frac{\partial^2 Y}{\partial t^2}=0, \quad -\infty<x<\infty,\ t>0, \tag{5.109}$$

$$Y(x,0)=F(x), \quad \frac{\partial Y}{\partial t}(x,0)=G(x), \quad -\infty<x<\infty. \tag{5.110}$$

Again suppose we want to make predictions in $0<t<T$. Then

$$|F(x)-f(x)|=0<\delta, \quad |G(x)-g(x)|=\delta\left|\cos\left(\frac{x}{\delta}\right)\right|<\delta, \tag{5.111}$$

and

$$|Y(0,t)-y(0,t)|=\delta^2\sinh\left(\frac{t}{\delta}\right)<\delta^2\sinh\left(\frac{T}{\delta}\right). \tag{5.112}$$

But

$$\delta^2\sinh\left(\frac{T}{\delta}\right)=\frac{1}{2}\delta^2\left(e^{T/\delta}-e^{-T/\delta}\right)\to\infty \quad \text{as} \quad \delta\to 0, \tag{5.113}$$

and we cannot make

$$|Y(0,t)-y(0,t)|<\epsilon \quad \text{for} \quad 0<t<T, \tag{5.114}$$

by making δ suitably small.