FOURIER SERIES AND PDEs
LECTURE 11

WAVES ON INFINITE STRINGS: D'ALEMBERT'S FORMULA

When the string occupies the interval $(-\infty, \infty)$ the fact that the solution has the form (4.100),

$$y(x,t) = F(x-ct) + G(x+ct), \tag{4.113}$$

is especially useful since we cannot now use separation of variables and Fourier series.

Consider the following IVP: find $y(x,t)$ if

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad \text{for } -\infty < x < \infty \text{ and } t > 0, \tag{4.114}$$

$$y(x,0) = f(x) \text{ and } \frac{\partial y}{\partial t}(x,0) = g(x) \text{ for } -\infty < x < \infty, \tag{4.115}$$

where f and g are prescribed functions.

To solve this we attempt to choose F and G in (4.113) so as to satisfy the initial conditions (4.115):

$$F(x) + G(x) = f(x), \quad -cF'(x) + cG'(x) = g(x). \tag{4.116}$$

The second integrates to give

$$-F(x) + G(x) = \frac{1}{c}\int_0^x g(s)\,\mathrm{d}s + a, \tag{4.117}$$

where a is a constant. Hence

$$F(x) = \frac{1}{2}\left[f(x) - \frac{1}{c}\int_0^x g(s)\,\mathrm{d}s - a\right], \tag{4.118}$$

$$G(x) = \frac{1}{2}\left[f(x) + \frac{1}{c}\int_0^x g(s)\,\mathrm{d}s + a\right], \tag{4.119}$$

and so

$$y(x,t) = \frac{1}{2}\left[f(x-ct) - \frac{1}{c}\int_0^{x-ct} g(s)\,\mathrm{d}s - a\right] + \frac{1}{2}\left[f(x+ct) + \frac{1}{c}\int_0^{x+ct} g(s)\,\mathrm{d}s + a\right]. \tag{4.120}$$

Thus we arrive at *D'Alembert's Formula*

$$y(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s. \tag{4.121}$$

The argument shows that, for given f and g, the IVP has one and only one solution (*i.e.* existence and uniqueness).

Next let us ask how the solution at a point $(x_0, t_0) = P$, in the upper half of the (x,t)-plane, depends upon the data f and g. We have

$$y(x_0,t_0) = \frac{1}{2}[f(x_0-ct_0) + f(x_0+ct_0)] + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} g(x)\,\mathrm{d}x, \tag{4.122}$$

Example 4.6 Find the solution of the wave equation for which

$$y(x,0) = 0, \quad -\infty < x < \infty, \tag{4.124}$$

$$\frac{\partial y}{\partial t}(x,0) = \begin{cases} 0 & x < -l, \\ vx/l & -l \le x \le l, \\ 0 & x > l. \end{cases} \tag{4.125}$$

In this case $g(x) = \partial y/\partial t(x,0)$ changes its analytic behaviour at the points $(-l,0)$ and $(l,0)$. We construct the characteristics through these points and thus divide up the upper-half of the (x,t)-plane into six regions $\mathbb{R}_1,\ldots,\mathbb{R}_6$, as shown ($\mathbb{R}_1$ is below $x+ct=-l$, $\mathbb{R}_2$ is above $x+ct=-l$ and above $x-ct=-l$, $\mathbb{R}_3$ is below $x-ct=-l$ and below $x+ct=l$, $\mathbb{R}_4$ is above $x+ct=l$ and above $x-ct=-l$, $\mathbb{R}_5$ is above $x+ct=l$ and above $x-ct=l$ and $\mathbb{R}_6$ is below $x-ct=l$).

In $\mathbb{R}_1$,

$$y(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} 0\,\mathrm{d}s = 0. \tag{4.126}$$

In $\mathbb{R}_2$,

$$y(x,t) = \frac{1}{2c}\int_{x-ct}^{-l} 0\,\mathrm{d}s + \frac{1}{2c}\int_{-l}^{x+ct} \frac{vs}{l}\,\mathrm{d}s = \frac{v}{4lc}\left[(x+ct)^2 - l^2\right]. \tag{4.127}$$

In $\mathbb{R}_3$,

$$y(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \frac{vs}{l}\,\mathrm{d}s = \frac{v}{4lc}\left[(x+ct)^2 - (x-ct)^2)\right] = \frac{vxt}{l}. \tag{4.128}$$

In $\mathbb{R}_4$,

$$y(x,t) = \frac{1}{2c}\int_{x-ct}^{-l} 0\,\mathrm{d}s + \frac{1}{2c}\int_{-l}^{l} \frac{vs}{l}\,\mathrm{d}s + \frac{1}{2c}\int_{l}^{x+ct} 0\,\mathrm{d}s = 0. \tag{4.129}$$

In $\mathbb{R}_5$,

$$y(x,t) = \frac{1}{2c}\int_{x-ct}^{l} \frac{vs}{l}\,\mathrm{d}s + \frac{1}{2c}\int_{l}^{x+ct} 0\,\mathrm{d}s = \frac{v}{4lc}\left[l^2 - (x-ct)^2\right]. \tag{4.130}$$

In $\mathbb{R}_6$,

$$y(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} 0\,\mathrm{d}s = 0. \tag{4.131}$$

From this information we can find the shape of the string. We illustrate this for a time $t > l/c$:

- for $x < -l - ct$ the point $(x,t) \in \mathbb{R}_1$ and $y(x,t) = 0$;
- for $-l - ct < x < l - ct$, $(x,t) \in \mathbb{R}_2$ and $y(x,t) = \frac{v}{4lc}\left[(x+ct)^2 - l^2\right]$;
- for $l - ct < x < -l + ct$, $(x,t) \in \mathbb{R}_4$ and $y(x,t) = 0$;
- for $-l + ct < x < l + ct$, $(x,t) \in \mathbb{R}_5$ and $y(x,t) = \frac{v}{4lc}\left[l^2 - (x-ct)^2\right]$;
- for $x > l + ct$, $(x,t) \in \mathbb{R}_6$ and $y(x,t) = 0$.

As t increases we see two packets of displacement, one moving to the left with speed c and the other to the right with speed c. Between them the displacement is zero.