UNIQUENESS OF AN IBVP FOR A FINITE STRING

We consider the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad \text{for } 0 < x < L \text{ and } t > 0, \tag{4.79}$$

and prove a uniqueness theorem based on energy considerations. The *kinetic energy* of the string is

$$\frac{1}{2}\int_0^L \rho \left(\frac{\partial y}{\partial t}\right)^2 \mathrm{d}x. \tag{4.80}$$

The *stress energy* is the product of the tension and the extension, where the extension is

$$\int_0^L \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}\,\mathrm{d}x - L = \int_0^L \left[1+\frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2+\ldots\right] \mathrm{d}x - L, \tag{4.81}$$

$$\approx \frac{1}{2}\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x. \tag{4.82}$$

Thus

$$E(t) := \frac{1}{2}\int_0^L \left[\rho\left(\frac{\partial y}{\partial t}\right)^2 + T\left(\frac{\partial y}{\partial x}\right)^2\right] \mathrm{d}x, \tag{4.83}$$

is the *energy* of the string. The energy appears to depend upon the time but in important cases it is actually constant.

Lemma 4.1 If $y(x,t)$ is a solution of the wave equation (4.79) and satisfies the boundary conditions

$$y(0,t) = 0 \text{ and } y(L,t) = 0 \text{ for } t \geq 0, \tag{4.84}$$

then $E(t)$ is constant for $t \geq 0$.

Proof. Leibniz's rule applied to equation (4.83) gives

$$E'(t) = \int_0^L \left[\rho \frac{\partial y}{\partial t}\frac{\partial^2 y}{\partial t^2} + T\frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial x \partial t}\right] \mathrm{d}x, \tag{4.85}$$

and, on substituting for $\rho\, \partial^2 y/\partial t^2$ from the wave equation (4.79), we find that

$$\begin{aligned} E'(t) &= T\int_0^L \left[\frac{\partial^2 y}{\partial x^2}\frac{\partial y}{\partial t} + \frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial x \partial t}\right] \mathrm{d}x, && (4.86)\\ &= T\int_0^L \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}\right) \mathrm{d}x, && (4.87)\\ &= T\left[\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}\right]_0^L, && (4.88) \end{aligned}$$

hence $E'(t) = 0$ since the boundary conditions (4.84) tell us that $\partial y/\partial t(0,t) = 0$ and $\partial y/\partial t(L,t) = 0$ for $t \geq 0$. Thus $E(t)$ is constant. □

Theorem 4.2 (Uniqueness) For each pair of functions f and g, the IBVP

$$\rho\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2}, \quad \text{for } 0 < x < L \text{ and } t > 0, \tag{4.89}$$

$$y(x,0) = f(x) \text{ and } \frac{\partial y}{\partial t}(x,0) = g(x) \text{ for } 0 \leq x \leq L, \tag{4.90}$$

$$y(0,t) = 0 \text{ and } y(L,t) = 0 \text{ for } t > 0, \tag{4.91}$$

has at most one solution. [That a solution exists we know since we have constructed a solution with the aid of separation of variables and Fourier series.]

Proof. Let $y(x,t)$ and $u(x,t)$ both be solutions of the IBVP. Consider the difference $w := y - u$ and the associated energy

$$E(t) := \frac{1}{2}\int_0^L \left[\rho\left(\frac{\partial w}{\partial t}\right)^2 + T\left(\frac{\partial w}{\partial x}\right)^2\right] \mathrm{d}x. \tag{4.92}$$

Note that $E \geq 0$. Now $w(x,t)$ is a solution of the IBVP

$$\rho\frac{\partial^2 w}{\partial t^2} = T\frac{\partial^2 w}{\partial x^2}, \quad \text{for } 0 < x < L \text{ and } t > 0, \tag{4.93}$$

$$w(x,0) = 0, \quad \frac{\partial w}{\partial t}(x,0) = 0 \quad (0 \leq x \leq L) \tag{4.94}$$

$$w(0,t) = 0 \text{ and } w(L,t) = 0 \quad (t > 0). \tag{4.95}$$

By Lemma 4.1 and the boundary conditions (4.95), $E(t) \equiv$ constant, and, in view of the initial conditions (4.94), $E(0) = 0$. Hence $E(t) = 0$ for every $t > 0$. Thus $\partial w/\partial x = 0$, $\partial w/\partial t = 0$ and $w(x,t)$ is independent of both x and t. Using (4.95) again tells us that $w(x,t) \equiv 0$. Thus $y = u$ and the solution is unique. □

The general solution of the wave equation

The wave equation is untypical among PDEs in that it is possible to write down all the solutions. Note that if $F : \mathbb{R} \to \mathbb{R}$ is any (twice differentiable) function, and

$$y(x,t) := F(x - ct), \tag{4.96}$$

then

$$\frac{\partial y}{\partial x} = F'(x - ct), \quad \frac{\partial^2 y}{\partial x^2} = F''(x - ct), \tag{4.97}$$

and

$$\frac{\partial y}{\partial t} = -cF'(x - ct), \quad \frac{\partial^2 y}{\partial t^2} = c^2 F''(x - ct), \tag{4.98}$$

and so (4.96) is a solution of the wave equation. Equation (4.96) represents a wave of constant shape propagating in the positive x-direction with speed c.

Similarly, if $G : \mathbb{R} \to \mathbb{R}$ is any (twice differentiable) function and

$$y(x,t) := G(x + ct), \tag{4.99}$$

then $y(x,t)$ is a solution of the wave equation. Equation (4.99) represents a wave of constant shape propagating in the negative x-direction with speed c.

Again, the sum

$$y(x,t) := F(x - ct) + G(x + ct), \tag{4.100}$$

is a solution of the wave equation. It will now be shown that *every solution of the wave equation must be of the form* (4.100).

To verify this introduce new independent variables

$$\xi := x - ct, \quad \eta := x + ct, \tag{4.101}$$

and seek a solution $y(x,t) = Y(\xi, \eta)$. Then

$$\frac{\partial y}{\partial x} = \frac{\partial Y}{\partial \xi} + \frac{\partial Y}{\partial \eta}, \quad \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 Y}{\partial \xi^2} + 2\frac{\partial^2 Y}{\partial \xi \partial \eta} + \frac{\partial^2 Y}{\partial \eta^2}, \tag{4.102}$$

$$\frac{\partial y}{\partial t} = -c\frac{\partial Y}{\partial \xi} + c\frac{\partial Y}{\partial \eta}, \quad \frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 Y}{\partial \xi^2} - 2c^2\frac{\partial^2 Y}{\partial \xi \partial \eta} + c^2\frac{\partial^2 Y}{\partial \eta^2}, \tag{4.103}$$

and substitution into the wave equation gives

$$\frac{\partial^2 Y}{\partial \xi^2} + 2\frac{\partial^2 Y}{\partial \xi \partial \eta} + \frac{\partial^2 Y}{\partial \eta^2} = \frac{\partial^2 Y}{\partial \xi^2} - 2\frac{\partial^2 Y}{\partial \xi \partial \eta} + \frac{\partial^2 Y}{\partial \eta^2}. \tag{4.104}$$

Hence in the new variables the wave equation transforms to the equation

$$\frac{\partial^2 Y}{\partial\xi\partial\eta} = 0, \tag{4.105}$$

i.e.

$$\frac{\partial}{\partial\xi}\left(\frac{\partial Y}{\partial\eta}\right) = 0. \tag{4.106}$$

Thus $\partial Y/\partial\eta$ is independent of ξ and is a function of η only, say $G'(\eta)$, *i.e.*

$$\frac{\partial Y}{\partial\eta} = G'(\eta), \tag{4.107}$$

and so

$$\frac{\partial}{\partial\eta}[Y - G(\eta)] = 0. \tag{4.108}$$

Thus $Y - G(\eta)$ is a function of ξ only, say $F(\xi)$, and therefore

$$Y - G(\eta) = F(\xi), \tag{4.109}$$

and

$$Y(\xi,\eta) = F(\xi) + G(\eta) \quad \Longrightarrow \quad y(x,t) = F(x-ct) + G(x+ct). \tag{4.110}$$

Further use of this conclusion will be made later.

Example 4.5 A string occupies $-\infty < x \leq 0$ and is fixed at $x = 0$. A wave $y(x,t) = F(x-ct)$ is incident from $x < 0$. Find the reflected wave.

The solution of the wave equation is

$$y = \underbrace{F(x-ct)}_{\text{incident}} + \underbrace{G(x+ct)}_{\text{reflected}}, \tag{4.111}$$

where G is to be found. The condition $y(0,t) = 0$ is to be satisfied for all t. Hence $F(-ct) + G(ct) = 0$, for all t, and so $G(\theta) = -F(-\theta)$ for all θ. Thus

$$y(x,t) = \underbrace{F(x-ct)}_{\text{incident}} - \underbrace{F(-x-ct)}_{\text{reflected}}. \tag{4.112}$$