APPLICATIONS OF **FOURIER SERIES**

To solve for more general initial conditions, we again look for a solution as a superposition of normal modes:

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right], \qquad (4.51)$$

so that we arrive at the problem: given $f(x)$ and $g(x)$ can they be expanded as Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right), \quad 0 \le x \le L, \qquad (4.52)$$

$$g(x) = \sum_{n=1}^{\infty} \left(\frac{n\pi c}{L}\right) b_n \sin\left(\frac{n\pi x}{L}\right), \quad 0 \le x \le L? \qquad (4.53)$$

From the lectures on Fourier Series we know that such an expansion as (4.52) exists if *e.g.* f and g are piecewise continuously differentiable on $[0, L]$. The coefficients are determined by the orthogonality relations:

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \begin{cases} 0 & m \neq n, \\ \frac{1}{2}L & m = n. \end{cases} \qquad (4.54)$$

Thus

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x, \tag{4.55}$$

$$b_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x. \tag{4.56}$$

Example 4.3 (Guitar or lute) For the special case (4.42),

$$a_n = \frac{2}{L}.\frac{2h}{L}\int_0^{\frac{L}{2}} x\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x + \frac{2}{L}.\frac{2h}{L}\int_{\frac{L}{2}}^L (L-x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x, \tag{4.57}$$

$$= \frac{8h}{\pi^2 n^2}\sin\left(\frac{n\pi}{2}\right), \tag{4.58}$$

and

$$b_n = \frac{2}{n\pi c}\int_0^L 0.\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x = 0. \tag{4.59}$$

Hence the solution is

$$y(x,t) = \frac{8h}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right), \tag{4.60}$$

$$= \frac{8h}{\pi^2}\left[\frac{1}{1^2}\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi ct}{L}\right) - \frac{1}{3^2}\sin\left(\frac{3\pi x}{L}\right)\cos\left(\frac{3\pi ct}{L}\right) + \frac{1}{5^2}\sin\left(\frac{5\pi x}{L}\right)\cos\left(\frac{5\pi ct}{L}\right) - \dots\right]. \tag{4.61}$$

Example 4.4 (Piano) The initial transverse displacement is zero and the section $[l_1, l_2]$ is given an initial transverse velocity v. Here $f(x) = 0$ for $0 \le x \le L$, and

$$g(x) = \begin{cases} 0 & \text{for } 0 \le x < L_1 \text{ and } L_2 < x \le L, \\ v & \text{for } L_1 \le x \le L_2. \end{cases} \tag{4.62}$$

Thus $a_n = 0$ and

$$b_n = \frac{2}{n\pi c}\int_{L_1}^{L_2} v\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x = \frac{2vL}{n^2\pi^2 c}\left[\cos\left(\frac{n\pi L_1}{L}\right) - \cos\left(\frac{n\pi L_2}{L}\right)\right]. \tag{4.63}$$

The transverse displacement is

$$y(x,t) = \frac{2vL}{\pi^2 c}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\cos\left(\frac{n\pi L_1}{L}\right) - \cos\left(\frac{n\pi L_2}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right). \tag{4.64}$$

Normal modes for a weighted string

A string of length $2L$ has fixed ends and a mass m is attached to the mid-point; find the normal frequencies of vibration.

Let the string occupy the interval $[-L, L]$, the mass being attached at $x = 0$. Let $y^-(x,t)$ and $y^+(x,t)$ be the transverse displacements for $-L \le x < 0$ and $0 < x \le L$, respectively. Then y^- and y^+ must satisfy the wave equations

$$\frac{\partial^2 y^-}{\partial t^2} = c^2 \frac{\partial^2 y^-}{\partial x^2}, \quad \frac{\partial^2 y^+}{\partial t^2} = c^2 \frac{\partial^2 y^+}{\partial x^2}, \tag{4.65}$$

and the boundary conditions

$$y^-(-L,t) = 0 \text{ and } y^+(L,t) = 0 \text{ for } t > 0. \tag{4.66}$$

What conditions hold at the mass m? There are two: firstly,

$$y^-(0,t) = y^+(0,t) \text{ for } t > 0; \tag{4.67}$$

and, secondly, the mass m is subject to Newton's Second Law,

$$m\frac{\partial^2 y}{\partial t^2}(0,t)\boldsymbol{j} = T\left(\boldsymbol{i} + \frac{\partial y^+}{\partial x}\boldsymbol{j}\right) - T\left(\boldsymbol{i} + \frac{\partial y^-}{\partial x}\boldsymbol{j}\right), \tag{4.68}$$

i.e.

$$m\frac{\partial^2 y}{\partial t^2}(0,t) = T\left[\frac{\partial y^+}{\partial x}(0,t) - \frac{\partial y^-}{\partial x}(0,t)\right] \text{ for } t > 0. \tag{4.69}$$

If we apply separation of variables arguments to (4.65) and (4.66) we see that y^-, y^+ must be of the form

$$y^-(x,t) = A\sin\left(\frac{\omega}{c}(L+x)\right)\cos(\omega t + \epsilon), \tag{4.70}$$

$$y^+(x,t) = B\sin\left(\frac{\omega}{c}(L-x)\right)\cos(\omega t + \epsilon), \tag{4.71}$$

where A, B, ϵ are constants and $\omega/(2\pi)$ is the, as yet unknown, normal frequency. Substitution into the boundary conditions (4.67) and (4.69) gives

$$A\sin\left(\frac{\omega L}{c}\right) = B\sin\left(\frac{\omega L}{c}\right), \tag{4.72}$$

and

$$-m\omega^2 A\sin\left(\frac{\omega L}{c}\right) = T\left[-B\left(\frac{\omega}{c}\right)\cos\left(\frac{\omega L}{c}\right) - A\left(\frac{\omega}{c}\right)\cos\left(\frac{\omega L}{c}\right)\right], \tag{4.73}$$

i.e.

$$A\left[\left(\frac{m\omega c}{T}\right)\sin\left(\frac{\omega L}{c}\right) - \cos\left(\frac{\omega L}{c}\right)\right] = B\cos\left(\frac{\omega L}{c}\right). \tag{4.74}$$

If the linear homogeneous equations (4.72) and (4.74) are to have non-trivial solutions (A and B not both zero) then the determinant must equal zero:

$$\sin\left(\frac{\omega L}{c}\right)\cos\left(\frac{\omega L}{c}\right) = \sin\left(\frac{\omega L}{c}\right)\left\{\left(\frac{m\omega c}{T}\right)\sin\left(\frac{\omega L}{c}\right) - \cos\left(\frac{\omega L}{c}\right)\right\}. \tag{4.75}$$

Thus either

$$\sin\left(\frac{\omega L}{c}\right) = 0 \quad \text{or} \quad \cot\left(\frac{\omega L}{c}\right) = \frac{2m\omega c}{T}. \tag{4.76}$$

The first equality in equation (4.76) implies

$$\frac{\omega L}{c} = n\pi \quad \text{i.e.} \quad \frac{\omega}{2\pi} = \frac{nc}{2L} = 2n.\frac{c}{2.2L}, \tag{4.77}$$

These correspond to the normal frequencies of a string of length $2L$ for which $x = 0$ is a node. There is no simple formula for the solutions of the other equality. If we put $\theta = \omega L/c$ then $\omega = c\theta/L$, where

$$\cot\theta = \frac{2mc^2}{TL}\theta = \frac{2m}{\rho L}\theta, \tag{4.78}$$

and we see there are infinitely many roots $\theta_1, \theta_2, \theta_3, \ldots$ and these determine infinitely many normal frequencies in addition to those given by (4.77).