FOURIER SERIES AND PDEs
LECTURE 8

NORMAL MODES OF VIBRATION FOR A FINITE STRING

A string is stretched between $x = 0$ and $x = L$ and the ends are held fixed. If the string is plucked, what notes do we hear? The question suggests that we want a solution which is periodic in time, with a period to be determined.

The displacement $y(x,t)$ satisfies the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \tag{4.24}$$

with boundary conditions

$$y(0,t) = 0 \text{ and } y(L,t) = 0 \text{ for } t > 0. \tag{4.25}$$

We use the separation of variables technique *i.e.* we attempt to find some (not all) solutions of equations (4.24) and (4.25) in the separable form

$$y(x,t) = F(x)G(t). \tag{4.26}$$

Substituting from (4.26) into (4.24) gives

$$\underbrace{c^2 \frac{F''(x)}{F(x)}}_{\text{independent of } t} = \underbrace{\frac{1}{G(t)} G''(t)}_{\text{independent of } x} . \tag{4.27}$$

Hence both sides are constant, independent of both x and t, and we take this constant to be equal to $-\omega^2$ to get

$$F''(x) = -\frac{\omega^2}{c^2} F(x), \quad G''(t) = -\omega^2 G. \tag{4.28}$$

The ODE for $F(x)$ is to be solved subject to the boundary conditions

$$F(0) = F(L) = 0. \tag{4.29}$$

We have

$$F(x) = A \sin\left(\frac{\omega x}{c}\right) + B \cos\left(\frac{\omega x}{c}\right), \tag{4.30}$$

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where the boundary condition at $x = 0$ gives $B = 0$, and the boundary condition at $x = L$ gives

$$A\sin\left(\frac{\omega L}{c}\right) = 0. \tag{4.31}$$

Since we want $A \neq 0$, otherwise $F = 0$ and $y = 0$, it must be that $\sin(\omega L/c) = 0$, *i.e.* ω must be such that $\omega L/c = n\pi$, where n is a positive integer, and ω must be one of the numbers

$$\left\{\frac{n\pi c}{L} : n = 1, 2, 3, \dots\right\}. \tag{4.32}$$

The ODE for $G(t)$ has the solution

$$G(t) = a\cos(\omega t) + b\sin(\omega t), \tag{4.33}$$

and so equations (4.24) and (4.25) have solutions of the form

$$y_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left[a_n\cos\left(\frac{n\pi ct}{L}\right) + b_n\sin\left(\frac{n\pi ct}{L}\right)\right], \tag{4.34}$$

where $n = 1, 2, 3, \dots$, and a_n and b_n are arbitrary constants. Such a solution is known as a *normal mode.*

A normal mode is periodic in t,

$$y(x, t+p) = y(x,t), \tag{4.35}$$

with *period*

$$p = \frac{2\pi}{\omega} = \frac{2L}{nc}, \tag{4.36}$$

and *frequency* (pitch)

$$\frac{1}{p} = \frac{\omega}{2\pi} = \frac{nc}{2L}. \tag{4.37}$$

The case $n = 1$ corresponds to the *fundamental frequency* $c/(2L)$, and all other normal frequencies are integer multiples of the fundamental frequency.

Note the graphs of the functions $\sin(n\pi x/L)$:

The general solution can be written as a super-position of normal modes:

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right], \tag{4.38}$$

and satisfies (4.24) and (4.25).

Initial-and-boundary value problems for finite strings

Consider the following IBVP: find $y(x,t)$ such that

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad \text{for } 0 < x < L \text{ and } t > 0, \tag{4.39}$$

and y satisfies the initial conditions

$$y(x,0) = f(x) \text{ and } \frac{\partial y}{\partial t}(x,0) = g(x) \text{ for } 0 \leq x \leq L, \tag{4.40}$$

and the boundary conditions

$$y(0,t) = 0 \text{ and } y(L,t) = 0 \text{ for } t > 0, \tag{4.41}$$

where $f(x)$ and $g(x)$ are known functions. According to (4.40), the initial transverse displacement and the initial transverse velocity are prescribed. If *e.g.*

$$f(x) = \begin{cases} 2hx/L & 0 \leq x \leq L/2, \\ 2h(L-x)/L & L/2 \leq x \leq L, \end{cases} \tag{4.42}$$

and $g(x) = 0$, $0 \leq x \leq L$, the mid-point of the string is pulled aside a distance h and the string is released from rest.

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and $g(x) = 0$, $0 \le x \le L$, the mid-point of the string is pulled aside a distance h and the string is released from rest.

We know that the general solution can be written as in equation (4.38), and hence the boundary conditions must satisfy

$$y(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right), \quad \frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} \left(\frac{n\pi c}{L}\right) b_n \sin\left(\frac{n\pi x}{L}\right). \tag{4.43}$$

Example 4.1 Solve the IBVP for the case

$$f(x) = A\sin\left(\frac{\pi x}{L}\right) + B\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right), \quad g(x) = 0. \tag{4.44}$$

Since

$$f(x) = A\sin\left(\frac{\pi x}{L}\right) + \frac{1}{2}B\sin\left(\frac{2\pi x}{L}\right), \tag{4.45}$$

the solution is obtained by taking $a_1 = A$, $a_2 = B/2$, $a_n = 0$ for $n \geq 3$ and $b_n = 0\ \forall n$ to give

$$y(x,t) = A\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi ct}{L}\right) + \frac{1}{2}B\sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{2\pi ct}{L}\right). \tag{4.46}$$

Example 4.2 Solve the IBVP for the case

$$f(x) = 0, \quad g(x) = \sin^3\left(\frac{\pi x}{L}\right). \tag{4.47}$$

Since

$$g(x) = \frac{3}{4}\sin\left(\frac{\pi x}{L}\right) - \frac{1}{4}\sin\left(\frac{3\pi x}{L}\right), \tag{4.48}$$

we take $a_n = 0\ \forall n$ and identify

$$\left(\frac{\pi c}{L}\right)b_1 = \frac{3}{4}, \quad \left(\frac{2\pi c}{L}\right)b_2 = 0, \quad \left(\frac{3\pi c}{L}\right)b_3 = -\frac{1}{4}, \quad b_n = 0 \text{ for } n \geq 4, \tag{4.49}$$

to arrive at

$$y(x,t) = \frac{3L}{4\pi c}\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi ct}{L}\right) - \frac{L}{12\pi c}\sin\left(\frac{3\pi x}{L}\right)\sin\left(\frac{3\pi ct}{L}\right). \tag{4.50}$$