WAVE EQUATION

Derivation in one space dimension

Consider a flexible string stretched to a tension T, with mass density ρ, undergoing small transverse vibrations. First suppose the string to be at rest along the x-axis in the (x, y)-plane. A point initially at $x\boldsymbol{i}$ is displaced to $\boldsymbol{r}(x,t) = x\boldsymbol{i} + y(x,t)\boldsymbol{j}$, where $y(x,t)$ is the transverse displacement and $\boldsymbol{i}$ and $\boldsymbol{j}$ are the usual unit vectors along the coordinate axes. We will assume that $|\partial y/\partial x| \ll 1$ and ignore gravity and air-resistance.

The vector

$$\boldsymbol{\tau} := \frac{\partial \boldsymbol{r}}{\partial x} = \boldsymbol{i} + \frac{\partial y}{\partial x}\boldsymbol{j}, \tag{4.1}$$

is a *tangent* vector to the string and, since

$$|\boldsymbol{\tau}| = \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} = 1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 - \frac{1}{8}\left(\frac{\partial y}{\partial x}\right)^4 + \cdots, \tag{4.2}$$

it is approximately a unit tangent. Thus, in the figure:

- $+T\boldsymbol{\tau}$ = force exerted by $+$ on $-$;
- $-T\boldsymbol{\tau}$ = force exerted by $-$ on $+$.

The *velocity* and *acceleration* vectors are

$$\boldsymbol{v} = \frac{\partial \boldsymbol{r}}{\partial t} = \frac{\partial y}{\partial t}\boldsymbol{j}, \qquad \boldsymbol{a} = \frac{\partial^2 \boldsymbol{r}}{\partial t^2} = \frac{\partial^2 y}{\partial t^2}\boldsymbol{j}, \tag{4.3}$$

respectively.

Consider the piece of string which occupies the interval $[a, a+h]$, where, at a later stage in the argument, $h \to 0$:

$$\text{net force} = T\boldsymbol{\tau}(a+h,t) - T\boldsymbol{\tau}(a,t); \tag{4.4}$$

$$\text{momentum} = \int_a^{a+h} \rho \boldsymbol{v}(x,t)\,\mathrm{d}x. \tag{4.5}$$

By *Newton's Second Law*, for every interval $[a, a+h]$,

$$\text{net force} = \text{rate of change of momentum}, \tag{4.6}$$

$$\implies T\boldsymbol{\tau}(a+h,t) - T\boldsymbol{\tau}(a,t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_a^{a+h} \rho \boldsymbol{v}(x,t)\,\mathrm{d}x. \tag{4.7}$$

On using Leibniz's rule, and dividing through by h, we see that

$$T\left(\frac{\boldsymbol{\tau}(a+h,t) - \boldsymbol{\tau}(a,t)}{h}\right) = \frac{1}{h}\int_a^{a+h} \rho \frac{\partial \boldsymbol{v}}{\partial t}(x,t)\,\mathrm{d}x, \tag{4.8}$$

and, on letting $h \to 0$, that

$$T\frac{\partial \boldsymbol{\tau}}{\partial x}(a,t) = \rho \frac{\partial \boldsymbol{v}}{\partial t}(a,t), \tag{4.9}$$

for every a. Thus, if we substitute for $\boldsymbol{\tau}$ and $\boldsymbol{v}$ in terms of the displacement y, we have

$$\rho\frac{\partial^2 y}{\partial t^2}\boldsymbol{j} = T\frac{\partial^2 y}{\partial x^2}\boldsymbol{j}, \tag{4.10}$$

and, hence, the wave equation

$$\rho\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2}, \tag{4.11}$$

or

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}, \tag{4.12}$$

where $c = \sqrt{T/\rho}$ is the *wave speed*.

Units and nondimensionalisation

Consider the units of the variables (x, t and y) and parameter (c) associated with the wave equation. For the wave equation,

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}, \tag{4.13}$$

we see that the left-hand side has units ms^{-2}. The term $\partial^2 y/\partial x^2$ has units m^{-1}. Hence for the units of the right-hand side to balance those of the left-hand side, the units of c must be $[c] = \mathrm{ms}^{-1}$, as we expect.

As before, can *non-dimensionalise* the wave equation by scaling our variables and parameters. For example, let

$$x = l\hat{x}, \quad t = \tau\hat{t}, \quad y = l\hat{y}, \tag{4.14}$$

where l, and τ are a typical lengthscale and timescale, respectively, for the problem under consideration. Then

$$\frac{\partial}{\partial t} = \frac{\mathrm{d}\hat{t}}{\mathrm{d}t}\frac{\partial}{\partial \hat{t}} = \frac{1}{\tau}\frac{\partial}{\partial \hat{t}}, \tag{4.15}$$

$$\frac{\partial^2}{\partial t^2} = \frac{\mathrm{d}\hat{t}}{\mathrm{d}t}\frac{\partial}{\partial \hat{t}}\left(\frac{1}{\tau}\frac{\partial}{\partial \hat{t}}\right) = \frac{1}{\tau^2}\frac{\partial^2}{\partial \hat{t}^2}, \tag{4.16}$$

$$\frac{\partial}{\partial x} = \frac{\mathrm{d}\hat{x}}{\mathrm{d}x}\frac{\partial}{\partial \hat{x}} = \frac{1}{l}\frac{\partial}{\partial \hat{x}}, \tag{4.17}$$

$$\frac{\partial^2}{\partial x^2} = \frac{\mathrm{d}\hat{x}}{\mathrm{d}x}\frac{\partial}{\partial \hat{x}}\left(\frac{1}{l}\frac{\partial}{\partial \hat{x}}\right) = \frac{1}{l^2}\frac{\partial^2}{\partial \hat{x}^2}, \tag{4.18}$$

and substituting into the wave equation we have

$$\frac{l}{\tau^2}\frac{\partial^2 \hat{y}}{\partial \hat{t}^2} = \frac{c^2 l}{l^2}\frac{\partial^2 \hat{y}}{\partial \hat{x}^2}. \tag{4.19}$$

Rearranging gives

$$\frac{\partial^2 \hat{y}}{\partial \hat{t}^2} = \frac{c^2\tau^2}{l^2}\frac{\partial^2 \hat{y}}{\partial \hat{x}^2}. \tag{4.20}$$

Considering the problem on a timescale where $\tau = l/c$ gives

$$\frac{\partial^2 \hat{y}}{\partial \hat{t}^2} = \frac{\partial^2 \hat{y}}{\partial \hat{x}^2}. \tag{4.21}$$

Notice that now

$$[\hat{x}] = 1, \quad [\hat{t}] = 1, \quad [\hat{y}] = 1, \tag{4.22}$$

since

$$[l] = \mathrm{m}, \quad [\tau] = \left[\frac{l}{c}\right] = \mathrm{s}. \tag{4.23}$$

This gives a relationship between problems with different lengthscales and wave speeds.