FOURIER SERIES AND PDEs
LECTURE 6

INITIAL AND BOUNDARY VALUE PROBLEM (IBVP)

We now consider finding the solution of the heat equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}, \quad 0 < x < L,\, t > 0, \tag{3.31}$$

subject to the initial condition

$$T(x,0) = f(x), \quad 0 \le x \le L, \tag{3.32}$$

and the boundary conditions

$$T(0,t) = 0 \text{ and } T(L,t) = 0 \text{ for } t > 0. \tag{3.33}$$

In view of our preceding discussion we look for a solution as an infinite sum

$$T(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2\kappa t/L^2}. \tag{3.34}$$

Example 3.1 Solve the IBVP for the case

$$T(x,0) = \sin\left(\frac{\pi x}{L}\right) + \frac{1}{2}\sin\left(\frac{2\pi x}{L}\right) = f(x), \quad 0 \le x \le L. \tag{3.35}$$

From equation (3.34) we see that

$$T(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right). \tag{3.36}$$

Comparing terms we see that $a_1 = 1$, $a_2 = 1/2$ and $a_n = 0$ $(n \ge 3)$ so that the solution is

$$T(x,t) = \sin\left(\frac{\pi x}{L}\right) e^{-\pi^2\kappa t/L^2} + \frac{1}{2}\sin\left(\frac{2\pi x}{L}\right) e^{-4\pi^2\kappa t/L^2}. \tag{3.37}$$

Application of Fourier series

To solve for more general initial conditions, we can use Fourier series to determine the constants a_n:

$$T(x,0)=\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right), \quad 0\le x\le L. \tag{3.38}$$

The question is now, given $f(x)$, can it be expanded as a Fourier sine series

$$f(x)=\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right), \quad 0\le x\le L? \tag{3.39}$$

From the lectures on Fourier series, we know that such an expansion exists if *e.g.* f is piecewise continuously differentiable on $[0,L]$. The coefficients a_n are determined by the orthogonality relations:

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x=\begin{cases}0, & m\ne n,\\ \frac{1}{2}L, & m=n.\end{cases} \tag{3.40}$$

Thus

$$a_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x. \tag{3.41}$$

Example 3.2 Find the solution of the IBVP when

$$f(x)=\begin{cases}0 & \text{for } 0\le x\le L_1 \text{ and } L_2\le x\le L,\\ 1 & \text{for } L_1<x<L_2.\end{cases} \tag{3.42}$$

Here $f(x)$ has the Fourier sine expansion

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\cos\left(\frac{n\pi L_1}{L}\right)-\cos\left(\frac{n\pi L_2}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right), \tag{3.43}$$

and the solution of IBVP is

$$T(x,t)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\cos\left(\frac{n\pi L_1}{L}\right)-\cos\left(\frac{n\pi L_2}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)e^{-n^2\pi^2\kappa t/L^2}. \tag{3.44}$$

Uniqueness

We have constructed a solution of our IBVP, and found a formula for it as the sum of an infinite series, but is it the only solution?

Theorem 3.1 The IBVP has only one solution.

Proof. Let U be a solution of the same IBVP, *i.e.*

$$\frac{\partial U}{\partial t} = \kappa \frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L,\, t > 0, \tag{3.45}$$

subject to the initial condition

$$U(x,0) = f(x), \quad 0 \le x \le L, \tag{3.46}$$

and the boundary conditions

$$U(0,t) = 0 \text{ and } U(L,t) = 0 \text{ for } t > 0. \tag{3.47}$$

Now consider the difference $W := U - T$. Then W satisfies the IBVP

$$\frac{\partial W}{\partial t} = \kappa \frac{\partial^2 W}{\partial x^2}, \quad 0 < x < L,\, t > 0, \tag{3.48}$$

$$W(x,0) = 0, \quad 0 \le x \le L, \tag{3.49}$$

and the boundary conditions

$$W(0,t) = 0 \text{ and } W(L,t) = 0 \text{ for } t > 0. \tag{3.50}$$

Let

$$I(t) := \frac{1}{2}\int_0^L [W(x,t)]^2 \, \mathrm{d}x. \tag{3.51}$$

Evidently $I(t) \ge 0$ and $I(0) = 0$. By Leibniz's rule,

$$\begin{aligned} I'(t) &= \int_0^L W \frac{\partial W}{\partial t} \, \mathrm{d}x, & (3.52)\\ &= \kappa \int_0^L W \frac{\partial^2 W}{\partial x^2} \, \mathrm{d}x, & (3.53)\\ &= \kappa \int_0^L \left[\frac{\partial}{\partial x}\left(W \frac{\partial W}{\partial x}\right) - \left(\frac{\partial W}{\partial x}\right)^2 \right] \mathrm{d}x. & (3.54) \end{aligned}$$

On carrying out the integration and using the boundary conditions at $x = 0$ and $x = L$ we see that

$$I'(t) = -\kappa \int_0^L \left(\frac{\partial W}{\partial x}\right)^2 \mathrm{d}x \le 0, \tag{3.55}$$

and, therefore, I cannot increase. Hence

$$0 \le I(t) \le I(0) = 0, \tag{3.56}$$

and $I(t) = 0$ for every $t \ge 0$. Thus

$$\int_0^L [W(x,t)]^2 \mathrm{d}x = 0, \tag{3.57}$$

for every $t \ge 0$ and so $W = 0$ and $U = T$, which proves the theorem. ☐

Non-zero steady state

It may be that the temperatures of the ends $x = 0$ and $x = L$ are prescribed and constant but not equal to zero.

Example 3.3 Solve the IBVP

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}, \quad 0 < x < L,\, t > 0, \tag{3.58}$$

subject to the initial condition

$$T(x, 0) = 0, \quad 0 \le x \le L, \tag{3.59}$$

and the boundary conditions

$$T(0, t) = T_0 \text{ and } T(L, t) = T_1 \text{ for } t > 0. \tag{3.60}$$

We cannot use separation of variables and Fourier series right at the outset. However, we conjecture that, as $t \to \infty$, $T(x, t) \to U(x)$, where

$$\kappa \frac{\mathrm{d}^2 U}{\mathrm{d}x^2} = 0, \quad U(0) = T_0 \text{ and } U(L) = T_1, \tag{3.61}$$

i.e.

$$U(x) = T_0 \left(1 - \frac{x}{L}\right) + T_1 \left(\frac{x}{L}\right). \tag{3.62}$$

If we now put $S(x, t) := T(x, t) - U(x)$, we find that S is a solution of the IBVP

$$\frac{\partial S}{\partial t} = \kappa \frac{\partial^2 S}{\partial x^2}, \quad 0 < x < L,\, t > 0, \tag{3.63}$$

with

$$S(0, t) = 0 \text{ and } S(L, t) = 0 \text{ for } t > 0, \tag{3.64}$$

and

$$S(x, 0) = -T_0 \left(1 - \frac{x}{L}\right) - T_1 \left(\frac{x}{L}\right). \tag{3.65}$$

In view of the form of the boundary conditions, this IBVP can be solved by our previous methods. The solution is

$$S(x, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[-T_0 + (-1)^n T_1\right] \sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2\kappa t/L^2}, \tag{3.66}$$

and so

$$T(x, t) = T_0 \left(1 - \frac{x}{L}\right) + T_1 \left(\frac{x}{L}\right) + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[-T_0 + (-1)^n T_1\right] \sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2\kappa t/L^2}. \tag{3.67}$$

Other boundary conditions

Other boundary conditions are possible, *e.g.* at an end which is *thermally insulated* the heat flux is zero. Thus $-kT_x = 0$ there and, therefore, $T_x = 0$. If both ends are thermally insulated we look for separable solutions of the heat equation of the form

$$T(x,t) = F(x)G(t), \tag{3.68}$$

where $F'(0) = F'(L) = 0$. We find that $F'' = -\lambda^2 F$, $G' = -\lambda^2 \kappa G$, and $F = a\cos(\lambda x)$, where $\sin(\lambda L) = 0$ and so L is one of the numbers $\{n\pi/L : n = 0, 1, 2, 3, \ldots\}$. The separable solutions in these circumstances are

$$a_0 \text{ and } a_n \cos\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2\kappa t/L^2} \quad (n = 1, 2, 3, \ldots). \tag{3.69}$$

Thus if we consider the IBVP

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}, \quad 0 < x < L,\, t > 0, \tag{3.70}$$

with boundary conditions

$$\frac{\partial T}{\partial x}(0,t) = 0 \text{ and } \frac{\partial T}{\partial x}(L,t) = 0 \text{ for } t > 0, \tag{3.71}$$

and initial condition

$$T(x,0) = f(x) \text{ for } 0 \le x \le L, \tag{3.72}$$

we look for a solution

$$T(x,t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2\kappa t/L^2}, \tag{3.73}$$

where the prescribed $f(x)$ has the Fourier cosine expansion

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \quad 0 \le x \le L. \tag{3.74}$$

The required coefficients are

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x \quad (n = 0, 1, 2, 3, \ldots). \tag{3.75}$$

Note that, as $t \to \infty$,

$$T(x,t) \to \frac{1}{2}a_0 = \frac{1}{L}\int_0^L f(s)\,\mathrm{d}s, \tag{3.76}$$

the extreme right-hand side being the mean initial temperature. The uniqueness of $T(x,t)$, for a given $f(x)$, can be established much as before.