DOOB'S DECOMPOSITION FOR SUPER-MARTINGALES

We introduce an important tool for the study of martingales, Doob's decomposition for square-integrable super-martingales. The extension to the continuous time case is much more difficult, called Doob-Meyer's decomposition, which is the key in order to define stochastic integrals with respect to martingales.

Suppose $X = (X_n)$ is a super-martingale on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_n, \mathbb{P})$. Thus $\mathbb{E}[X_{n+1}|\mathcal{F}_n] \leq X_n$, so roughly speaking on average, $n \to X_n$ is decreasing. Doob's decomposition is an explicit statement about this fact. The idea is to seek for a martingale M_n and an increasing sequence A_n such that $X_n = M_n - A_n$. Let $A_0 = 0$ and $M_0 = X_0$. Since

$$X_{n+1} - X_n = M_{n+1} - M_n - (A_{n+1} - A_n)$$

and conditional on $\mathcal{F}_n$, to obtain

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] - X_n = -\mathbb{E}[A_{n+1} - A_n|\mathcal{F}_n].$$

If we impose the condition that A_n is $\mathcal{F}_{n-1}$-measurable for every $n \geq 1$ (such a sequence is called predictable). Then

$$A_{n+1} = A_n + X_n - \mathbb{E}[X_{n+1}|\mathcal{F}_n] = \sum_{j=0}^{n} (X_j - \mathbb{E}[X_{j+1}|\mathcal{F}_j]) = \sum_{j=0}^{n} \mathbb{E}[X_j - X_{j+1}|\mathcal{F}_j]$$

for $n \geq 0$. We note that, since X_n is a super-martingale, thus (A_n) is increasing and predictable, with $A_0 = 0$, and it is direct to verify that $M_n = X_n + A_n$ is a martingale.

Theorem 13.1 *(Doob's decomposition for super-martingales) Let $X = (X_n)$ be a super-martingale over a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_n, \mathbb{P})$. Then there is a unique increasing predictable random sequence (A_n) with $A_0 = 0$, such that $M_n = X_n + A_n$ is a martingale. More precisely*

$$A_n = \sum_{j=0}^{n-1} \left(X_j - \mathbb{E}\left[X_{j+1}|\mathcal{F}_j\right]\right)$$

for $n \geq 1$, and

$$M_n = X_n + \sum_{j=0}^{n-1} \left(X_j - \mathbb{E}\left[X_{j+1}|\mathcal{F}_j\right]\right)$$

for $n = 1, 2, \cdots$, and $A_0 = 0$, $M_0 = X_0$. The decomposition $X_n = M_n - A_n$ is called Doob's decomposition for the super-martingale $X = (X_n)$.

Let us apply Doob's decomposition to square integrable martingales.

Suppose that $M = (M_n)$ is a martingale such that $\mathbb{E}[M_n^2] < \infty$ for each n. Then M_n^2 is a sub-martingale, so $-M_n^2$ is a super-martingale. Therefore there is a unique increasing predictable random sequence A_n such that $-M_n^2 + A_n$ is again a martingale, where

$$\begin{aligned} A_n &= \sum_{j=0}^{n-1} \left(-M_j^2 + \mathbb{E}\left[M_{j+1}^2|\mathcal{F}_j\right]\right) = \sum_{j=0}^{n-1} \mathbb{E}\left[M_{j+1}^2 - M_j^2|\mathcal{F}_j\right] \\ &= \sum_{j=0}^{n-1} \mathbb{E}\left[\left(M_{j+1} - M_j\right)^2 |\mathcal{F}_j\right] \end{aligned}$$

which is called the *bracket process* associated with M. The bracket process will play an important role in the study of martingales, so let us give a definition.

Definition 13.2 *1) Let $M = (M_n)$ be a* martingale *with $M_n \in L^2(\Omega)$ for every n. Then the bracket process $\langle M \rangle$ associated with M is the unique predictable, increasing sequence with $\langle M \rangle_0 = 0$ such that $M_n^2 - \langle M \rangle_n$ is a martingale. Explicitly $\langle M \rangle$ is given by*

$$\langle M \rangle_n = \sum_{j=0}^{n-1} \mathbb{E}\left[(M_{j+1} - M_j)^2|\mathcal{F}_j\right]$$

for $n \geq 1$, $\langle M \rangle_0 = 0$. That is, $\langle M \rangle$ is the conditional quadratic variation process associated with M. In particular, for any bounded *stopping time T, $\mathbb{E}[M_T^2 - M_0^2] = \mathbb{E}[\langle M \rangle_T]$, and*

$$\sup_n \mathbb{E}\left[M_n^2 - M_0^2\right] = \sup_n \mathbb{E}\left[\langle M \rangle_n\right] = \lim_{n\to\infty} \mathbb{E}\left[\langle M \rangle_n\right] = \mathbb{E}\left[\langle M \rangle_\infty\right]$$

where $\langle M \rangle_\infty = \lim_{n\to\infty} \langle M \rangle_n$ (which may be infinity), and the last equality follows from MCT applying to $\langle M \rangle_n \uparrow \langle M \rangle_\infty$.

2)The quadratic variation process $[M]_n$ associated with M is defined by $[M]_0 = 0$ and

$$[M]_n = \sum_{j=0}^{n-1} (M_{j+1} - M_j)^2$$

for $n \geq 1$.

3) A martingale $M = (M_n)$ is called a squared integrable martingale if $\sup_n \mathbb{E}[M_n^2] < \infty$ (i.e. $\{M_n : n \geq 0\}$ is bounded in $L^2(\Omega)$.)

By a direct computation we have

Lemma 13.3 *1) Let $M = (M_n)$ be a martingale on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_n, \mathbb{P})$ such that $M_n \in L^2(\Omega)$. Then $[M]_n - \langle M \rangle_n$ is a martingale, while $\langle M \rangle$ is predictable, and $[M]$ is an adapted increasing sequence.*

2) Suppose that M and N are two martingales such that $M_n, N_n \in L^2(\Omega)$, then $M_n N_n - \langle M, N \rangle_n$ is a martingale, where the mutual bracket

$$\begin{aligned}\langle M, N \rangle_n &= \frac{1}{4}\left(\langle M + N \rangle - \langle M - N \rangle\right) \\ &= \sum_{j=0}^{n-1} \mathbb{E}\left[(M_{j+1} - M_j)(N_{j+1} - N_j) | \mathcal{F}_j\right].\end{aligned}$$

for $n \geq 1$, $\langle M, N \rangle_0 = 0$, which is a predictable process.

Suppose that $M = (M_n)$ is a martingale, and $H = (H_n)$ is a predictable process, the martingale transform $H.M$ (which corresponds the Ito integral of H against M, so called discrete stochastic integral of H against M) is defined by $(H.M)_0 = 0$ and

$$(H.M)_n = \sum_{j=1}^{n} H_j (M_j - M_{j-1})$$

for $n \geq 1$. Then

$$\begin{aligned}\langle H.M \rangle_n &= \sum_{j=0}^{n-1} \mathbb{E}\left[(H_{j+1}(M_{j+1} - M_j))^2 | \mathcal{F}_j\right] = \sum_{j=0}^{n-1} H_{j+1}^2 \mathbb{E}\left[(M_{j+1} - M_j)^2 | \mathcal{F}_j\right] \\ &= \sum_{j=1}^{n} H_j^2 (\langle M \rangle_j - \langle M \rangle_{j-1})\end{aligned}$$

which is $H^2.\langle M \rangle$, the stochastic integral of H^2 with respect to the increasing process $\langle M \rangle$.

The bracket processes play a fundamental role in Ito's stochastic integration theory. Here we only give an elementary application of the bracket process.

Theorem 13.4 *Let $M = (M_n)$ be a martingale on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_n, \mathbb{P})$ such that $M_n \in L^2(\Omega)$. Then $M_\infty = \lim_{n \to \infty} M_n$ exists on $\{\langle M \rangle_\infty < \infty\}$.*

Proof. [*The proof is not examinable*]. Since

$$\{\langle M \rangle_\infty < \infty\} = \bigcup_{l=1}^{\infty} \{\langle M \rangle_\infty \leq l\}$$

so we only need to show that $M_\infty = \lim_{n \to \infty} M_n$ exists on each $\{\langle M \rangle_\infty \leq l\}$. Let $l > 0$, and $T = \inf\{k \geq 0 : \langle M \rangle_{k+1} > l\}$. Then T is a stopping time as $\langle M \rangle$ is predictable, so that by Theorem 10.4, $M_{T\wedge n}^2 - \langle M \rangle_{T \wedge n}$ is martingale, thus $\mathbb{E}\left[M_{T\wedge n}^2\right] = \mathbb{E}\left[\langle M \rangle_{T\wedge n}\right] \leq l$ for all n. Therefore $\{M_{T\wedge n}\}$ is a uniformly integrable martingale, so that $\lim_{n\to\infty} M_{T\wedge n}$ exists. In particular, $\lim_{n\to\infty} M_n$ exists on $\{T = \infty\}$, so does on $\{\langle M \rangle_\infty \leq l\}$ for any $l > 0$. ∎