PROBABILITY LIMIT THEOREM

The martingale convergence theorem is a powerful tool to show the limit theorems for random sequences.

12.1 Levy's upward and downward

Corollary 12.1 (Levy's "Upward" theorem) *Let $\xi \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}_n)_{n\geq 0}$ is an increasing family of sub σ-algebras of $\mathcal{F}$. Then*

$$\lim_{n\to\infty} \mathbb{E}[\xi|\mathcal{F}_n] = \mathbb{E}[\xi|\mathcal{F}_\infty] \text{ almost surely and in } L^1(\Omega)$$

where $\mathcal{F}_\infty = \sigma\{\mathcal{F}_j : j \geq 0\}$.

Proof. Let $X_n = \mathbb{E}[\xi|\mathcal{F}_n]$ for $n \geq 0$. Since $\xi \in L^1(\Omega)$, $X = (X_n)$ is a uniformly integrable martingale, so by Doob's martingale convergence theorem, $X_n \to X_\infty$, for some $X_\infty \in L^1(\Omega)$, almost everywhere and in $L^1(\Omega)$. We need to show that $X_\infty = \mathbb{E}[\xi|\mathcal{F}_\infty]$. By considering ξ^+ and ξ^- instead, we may assume that ξ is non-negative. Thus $X_\infty \geq 0$ a.e. and X_∞ is $\mathcal{F}_\infty$-measurable. To show that $X_\infty = \mathbb{E}[\xi|\mathcal{F}_\infty]$ we only need to prove that for every $A \in \mathcal{F}_\infty$, $\mathbb{E}[X_\infty : A] = \mathbb{E}[\xi : A]$. To this end we consider two finite measures $\mu_1(B) = \mathbb{E}[X_\infty : B]$ and $\mu_2(B) = \mathbb{E}[\xi : B]$ for $B \in \mathcal{F}$. What we need to prove is that $\mu_1 = \mu_2$ on $\mathcal{F}_\infty$. To this end we can utilize the Uniqueness Lemma for finite measures, Lemma 2.2 or Dynkin's lemma.

Let $\mathcal{C} = \cup_{j=0}^{\infty}\mathcal{F}_j$ which is a π-system, and let

$$\begin{aligned}\mathcal{G} &= \{B \in \mathcal{F} : \mu_1(B) = \mu_2(B)\} \\ &= \{B \in \mathcal{F} : \mathbb{E}[X_\infty : B] = \mathbb{E}[\xi : B]\}.\end{aligned}$$

If $A \in \mathcal{C}$, then there is n, such that $A \in \mathcal{F}_n \subseteq \mathcal{F}_m$ for all $m \geq n$. Hence

$$\mathbb{E}[X_m : A] = \mathbb{E}[\xi : A]$$

and by letting $m \uparrow \infty$ we obtain that we have $\mathbb{E}(1_A X_\infty) = \mathbb{E}(1_A \xi)$. Hence $\mathcal{C} \subseteq \mathcal{G}$. Suppose $A_n \in \mathcal{G}$ and $A_n \uparrow A$, then $X_\infty 1_{A_n} \uparrow X_\infty 1_A$ and $\xi 1_{A_n} \uparrow \xi 1_A$ as $n \uparrow \infty$. By MCT and the assumption that $\mathbb{E}[X_\infty : A_n] = \mathbb{E}[\xi : A_n]$ for every n, we conclude that

$$\mathbb{E}[X_\infty : A] = \lim_{n\to\infty} \mathbb{E}[X_\infty : A_n] = \lim_{n\to\infty} \mathbb{E}[\xi : A_n] = \mathbb{E}[\xi : A]$$

so that $A \in \mathcal{G}$ too. Thus $\mathcal{G}$ is a monotone class, containing the π-system $\mathcal{C}$. Hence, by Dynkin's lemma, $\mathcal{G} \supseteq \sigma\{\mathcal{C}\} = \mathcal{F}_\infty$, which implies that $\mu_1 = \mu_2$ on $\mathcal{F}_\infty$. Therefore $X_\infty = \mathbb{E}[\xi|\mathcal{F}_\infty]$. The proof is complete. ■

Corollary 12.2 *(*Levy's "Downward" theorem*) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space together with a decreasing family $(\mathcal{G}_n)_{n\geq 0}$ of sub σalgebras of $\mathcal{F}$: $\mathcal{G}_{n+1} \subseteq \mathcal{G}_n$ for every n. Let $\xi \in L^1(\Omega)$. Then*

$$\lim_{n\to\infty} \mathbb{E}[\xi|\mathcal{G}_n] = \mathbb{E}\left[\xi|\bigcap_{j=0}^{\infty} \mathcal{G}_j\right] \text{ almost surely and in } L^1(\Omega).$$

This follows from the downward martingale convergence theorem.

12.2 Kolmogorov's 0-1 law

Corollary 12.3 *(*Kolmogorov's 0-1 law*) Let ξ_n $(n = 1, 2, \cdots)$ be a sequence of independent random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{G}_n = \sigma\{\xi_j : j \geq n+1\}$ and $\mathcal{G}_\infty = \cap_{n=0}^{\infty} \mathcal{G}_n$. An element in $\mathcal{G}_\infty$is called a tail event. If Z is $\mathcal{G}_\infty$-measurable and integrable, then $Z = \mathbb{E}(Z)$ almost surely. Thus any $\mathcal{G}_\infty$-measurable random variable is constant almost surely.*

Proof. Let $\mathcal{F}_n = \sigma\{\xi_j : j \leq n\}$. Then, for every n, $\mathcal{F}_n$ and $\mathcal{G}_n$ are independent. Hence $\mathcal{F}_n$ and $\mathcal{G}_\infty$ are independent for any n. Let $X_n = \mathbb{E}[Z|\mathcal{F}_n]$. Then $X = (X_n)$ is a uniformly integrable martingale, so that $X_n \to \mathbb{E}[Z|\mathcal{F}_\infty]$ almost surely and in $L^1(\Omega)$. While $\mathcal{F}_\infty \supset \mathcal{G}_\infty$, so that $\mathbb{E}[Z|\mathcal{F}_\infty] = Z$ a.e. and therefore $X_n \to Z$ almost surely and in L^1. On the other hand, since Z and $\mathcal{F}_n$ are independent, so that $X_n = \mathbb{E}[Z|\mathcal{F}_n] = \mathbb{E}[Z]$ almost everywhere. Therefore $Z = \mathbb{E}(Z)$ almost surely. ■

12.3 The strong law of large numbers

Let us prove the *strong law of large numbers* for i.i.d. sequences.

Theorem 12.4 *(A. Kolmogorov,* The Strong Law of Large Numbers*) Let $\{\xi_k\}_{k\geq 1}$ be a sequence of independent integrable random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with the same distribution. Then $\frac{1}{n}\sum_{k=1}^{n} \xi_k \to \mathbb{E}[\xi_1]$ almost everywhere.*

Proof. Let $(\mathcal{G}_n)_{n\geq 0}$ be the decreasing family of σ-algebras generated by the sequence (X_n), where $X_n = \sum_{k=1}^{n} \xi_k$. That is

$$\mathcal{G}_n = \sigma\{X_m : m \geq n\} = \sigma\{X_n, \xi_j \geq n+1\}.$$

Since $\xi_1, \cdots, \xi_n$ are independent with the same distribution, so that

$$X_n = \sum_{i=1}^{n} \mathbb{E}\left[\xi_i | \xi_1 + \cdots + \xi_n\right] = n\mathbb{E}\left[\xi_1 | \xi_1 + \cdots + \xi_n\right],$$

which implies that

$$\begin{aligned}\frac{X_n}{n} &= \mathbb{E}\left[\xi_1 | X_n\right] = \mathbb{E}\left[\xi_1 | X_n, \xi_j \text{ for } j \geq n+1\right] \\ &= \mathbb{E}\left[\xi_1 | \mathcal{G}_n\right]\end{aligned}$$

for every n. Thus according to Levy's downward theorem

$$\frac{X_n}{n} \to \mathbb{E}\left[\xi_1 | \bigcap_{n=1}^{\infty} \mathcal{G}_n\right] = \mathbb{E}\left[\xi_1 | \mathcal{G}_\infty\right]$$

almost surely and in L^1, where $\mathcal{G}_\infty$ is the tail σ-algebra. According to Kolmogorov's 0-1 law, since $\mathbb{E}\left[\xi_1 | \mathcal{G}_\infty\right]$ is $\mathcal{G}_\infty$-measurable, $\mathbb{E}\left[\xi_1 | \mathcal{G}_\infty\right] = \mathbb{E}\left[\xi_1\right]$ almost surely. Therefore

$$\frac{X_n}{n} \to \mathbb{E}\left[\xi_1\right]$$

almost surely and in $L^1(\Omega)$. ■

We should point out that the strong law of large numbers for i.i.d. sequences is still a special case of *Birkhoff's ergodic theorem* for strictly stationary sequences. Birkhoff's ergodic theorem however a different approach and thus provides a different proof for the strong law of large numbers.