THE MARTINGALE CONVERGENCE THEOREM

An important field in the probability theory is to study the asymptotic behavior of sequences of random variables. For example, we are interested in whether a sequence $\{X_n : n \geq 0\}$ converges or not as $n \to \infty$.

11.1 Doob's up-crossing lemma

One of the powerful tools to study the convergence of random sequences is the concept of up-crossing numbers through intervals by a random sequence.

Suppose (a_n) is a sequence of real numbers, then $\lim_{n\to\infty} a_n$ exists (as a real number), $\lim_{n\to\infty} a_n = \infty$ or $\lim_{n\to\infty} a_n = -\infty$, if and only if $\liminf a_n = \limsup a_n$. Therefore, if (X_n) is a random sequence of real random variables, then

$$\lim_{n\to\infty} a_n \text{ exists in } [-\infty, \infty] \text{ if and only if } \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n.$$

Moreover, by definition, there are two sub-sequences n_k and m_k such that

$$\lim_{k\to\infty} a_{n_k} = \liminf_{n\to\infty} a_n \text{ and } \lim_{l\to\infty} a_{m_l} = \limsup_{n\to\infty} a_n$$

where we can choose two sub-sequences such that

$$n_0 < m_0 < n_1 < m_1 < \cdots < n_k < m_k < \cdots$$

In the case that $\liminf_{n\to\infty} a_n < \limsup_{n\to\infty} a_n$, then we can choose $a < b$ such that

$$\liminf_{n\to\infty} a_n < a < b < \limsup_{n\to\infty} a_n$$

(and we can demand that $a < b$ to be rational numbers). Then, by looking the sequence (a_n) along $a_{n_0}, a_{m_0}, \cdots, a_{n_k}, a_{m_k}, \cdots$, we can see that the sequence (a_n) must cross from low level a to upper level b infinitely many times. That is, the number of up-crossing (a, b) by (a_n) is infinite. Hencelim$_{n\to\infty} a_n$ exists in $[-\infty, \infty]$ if and only if the *up-crossing number* by (a_n) through any (a, b) (for every pair $a < b$ of rational numbers) is finite.

Apply this to a sequence (X_n) of real random variables,

$$\left\{\lim_{n\to\infty} X_n \text{ exists in } [-\infty, \infty]\right\} = \{\text{the up-corssing number of } (X_n) \text{ through } (a, b) < \infty \text{ for any rationals}$$

Let $X = (X_n)_{n\geq 0}$ be a sequence of real valued random variables, and $a < b$ be two numbers. An *up-crossing* is the event that the sequence (X_n) is below a at some n and then $X_m \geq b$ for some $m > n$, and similarly we may define a down-crossing. Let us concentrate on up-crossing events.

Define

$$\begin{aligned} T_0 &= \inf\{n \geq 0 : X_n \leq a\}, \\ T_1 &= \inf\{n > T_0 : X_n \geq b\}, \\ &\cdots\cdots \\ T_{2j} &= \inf\{n > T_{2j-1} : X_n \leq a\}, \\ T_{2j+1} &= \inf\{n > T_{2j} : X_n \geq b\}, \\ &\cdots \end{aligned}$$

T_0 is the first time that the sequence X goes to the level below a, and T_1 is the first time X goes back to the level b after reaching the level below a and so on. All T_k are random times but can take value ∞, and $\{T_k\}$ is increasing. Moreover

$$\begin{aligned} X_{T_{2j}} &\leq a \quad \text{on} \quad \{T_{2j} < \infty\}, \\ X_{T_{2j+1}} &\geq b \quad \text{on} \quad \{T_{2j+1} < \infty\}. \end{aligned}$$

If $T_{2j-1}(\omega) < \infty$ for some $j \in \mathbb{N}$, then the sequence

$$X_0(\omega), \cdots, X_{T_{2j-1}}(\omega)$$

up-crosses the interval $[a, b]$ exactly j times.

Let $U_a^b(X; n)$ denote the number of up-crossings of $\{X_0, \cdots, X_n\}$ through interval $[a, b]$. Then

$$\left\{U_a^b(X; n) = j\right\} \subset \{T_{2j-1} \leq n < T_{2j+1}\} \tag{11.1}$$

and

$$\left\{U_a^b(X; n) \geq j\right\} = \{T_{2j-1} \leq n\} \tag{11.2}$$

for $j = 0, 1, \cdots$.

If $X = (X_n)_{n \geq 0}$ is adapted with respect to a filtration $\{\mathcal{F}_n : n \geq 0\}$, then T_k are stopping times. Hence

$$\left\{U_a^b(X; n) = j\right\} = \{T_{2j-1} \leq n\} \bigcap \{T_{2j=1} > n\} \in \mathcal{F}_n$$

for any $n \in \mathbb{Z}_+$ and $j \in \bar{\mathbb{Z}}_+$.

Lemma 11.1 *For any $b > a$ and $n, k \in \mathbb{N}$ we have*

$$1_{\left\{U_a^b(X;n)\geq k\right\}} \leq -\frac{X_n - a}{b-a} 1_{\left\{T_{2(k-1)}\leq n<T_{2k-1}\right\}} + \frac{X_{T_{2k-1}\wedge n} - X_{T_{2(k-1)}\wedge n}}{b-a} \tag{11.3}$$

and

$$1_{\left\{U_a^b(X;n)\geq k\right\}} \leq \frac{X_n - a}{b-a} 1_{\{T_{2k-1}\leq n<T_{2k}\}} + \frac{X_{T_{2k-1}\wedge n} - X_{T_{2k}\wedge n}}{b-a}. \tag{11.4}$$

Proof. [*The proof is not examinable*] For every $k = 1, 2, \cdots$, $T_{2(k-1)} < T_{2k-1} < T_{2k}$ on $\{T_{2k-1} < \infty\}$. Let us consider the increments of $X = (X_n)$ over $[T_{2k-2}, T_{2k-1}]$ and $[T_{2k-1}, T_{2k}]$ respectively, which must be greater than $b - a$ on $\{T_{2k-1} < \infty\}$ (resp. on $\{T_{2k} < \infty\}$).

It is elementary that

$$\begin{aligned} X_{T_{2k-1}\wedge n} - X_{T_{2(k-1)}\wedge n} &= \left(X_{T_{2k-1}\wedge n} - X_{T_{2(k-1)}}\right) 1_{\left\{T_{2(k-1)}\leq n\right\}} \\ &= \left(X_{T_{2k-1}} - X_{T_{2(k-1)}}\right) 1_{\left\{T_{2(k-1)}\leq n\right\}} 1_{\{T_{2k-1}\leq n\}} \\ &\quad + \left(X_n - X_{T_{2(k-1)}}\right) 1_{\left\{T_{2(k-1)}\leq n\right\}} 1_{\{T_{2k-1}>n\}} \\ &= \left(X_{T_{2k-1}} - X_{T_{2(k-1)}}\right) 1_{\{T_{2k-1}\leq n\}} \\ &\quad + \left(X_n - X_{T_{2(k-1)}}\right) 1_{\{T_{2(k-1)}\leq n<T_{2k-1}\}}. \end{aligned}$$

Since $X_{T_{2k-1}} - X_{T_{2(k-1)}} \geq b - a$ on $\{T_{2k-1} < \infty\}$, $X_{T_{2(k-1)}} \leq a$ on $\left\{T_{2(k-1)} < \infty\right\}$, and since $\{T_{2k-1} \leq n\} = \left\{U_a^b(X; n) \geq k\right\}$, we deduce from the previous identity that

$$X_{T_{2k-1}\wedge n} - X_{T_{2(k-1)}\wedge n} \geq (b-a)1_{\left\{U_a^b(X;n)\geq k\right\}} + (X_n - a)\, 1_{\left\{T_{2(k-1)}\leq n<T_{2k-1}\right\}}$$

and (11.3) follows. Similarly, one may use the decomposition

$$\begin{aligned} X_{T_{2k-1}\wedge n} - X_{T_{2k}\wedge n} &= \left(X_{T_{2k-1}} - X_{T_{2k}}\right) 1_{\{T_{2k}\leq n\}} + \left(X_{T_{2k-1}} - X_n\right) 1_{\{T_{2k-1}\leq n<T_{2k}\}} \\ &\geq (b-a)\, 1_{\{T_{2k}\leq n\}} + (b - X_n)\, 1_{\{T_{2k-1}\leq n<T_{2k}\}} \\ &= (b-a)\left(1_{\{T_{2k}\leq n\}} + 1_{\{T_{2k-1}\leq n<T_{2k}\}}\right) + (a - X_n)\, 1_{\{T_{2k-1}\leq n<T_{2k}\}} \\ &= (b-a)\, 1_{\{T_{2k-1}\leq n\}} + (a - X_n)\, 1_{\{T_{2k-1}\leq n<T_{2k}\}} \end{aligned}$$

where we have used the fact that $X_{T_{2k-1}} \geq b$ and $X_{T_{2k}} \leq a$ on $\{T_{2k} < \infty\}$, which yields that

$$1_{\{T_{2k-1}\leq n\}} \leq -\frac{a - X_n}{b-a} 1_{\{T_{2k-1}\leq n<T_{2k}\}} + \frac{X_{T_{2k-1}\wedge n} - X_{T_{2k}\wedge n}}{b-a}.$$

■

Theorem 11.2 *(*Doob's up-crossing lemma*) 1) If $X = (X_n)$ is a super-martingale, then for any $n \geq 1$, $k \geq 1$*

$$\mathbb{P}\left[U_a^b(X;n) \geq k\right] \leq \mathbb{E}\left[\frac{(X_n - a)^-}{b-a} : U_a^b(X;n) = k-1\right]$$

and

$$\mathbb{E}\left[U_a^b(X;n)\right] \leq \mathbb{E}\left[\frac{(X_n - a)^-}{b-a}\right].$$

[Note that $X_n - a$ is also a super-martingale for any constant a, so that $(X_n - a)^-$ is a sub-martingale.*]*

2) Similarly, if $X = (X_n)$ is a sub-martingale, then

$$\mathbb{P}\left[U_a^b(X;n) \geq k\right] \leq \mathbb{E}\left[\frac{(X_n - a)^+}{b-a} : U_a^b(X;n) = k\right]$$

and

$$\mathbb{E}\left[U_a^b(X;n)\right] \leq \mathbb{E}\left[\frac{(X_n - a)^+}{b-a}\right].$$

[For a sub-martingale, $(X_n - a)^+$ is again a sub-martingale for every constant a.]

Proof. [*The proof is not examinable*] 1) Since X is a super-martingale, according to Doob's optional stopping theorem

$$\mathbb{E}\left[X_{T_{2k-1}\wedge n} - X_{T_{2(k-1)}\wedge n}\right] \leq 0, \tag{11.5}$$

so that it follows from (11.3) that

$$\begin{aligned}
\mathbb{P}\left[U_a^b(X;n) \geq k\right] &\leq -\mathbb{E}\left[\frac{X_n - a}{b-a} : T_{2(k-1)} \leq n < T_{2k-1}\right] + \mathbb{E}\left[X_{T_{2k-1}\wedge n} - X_{T_{2(k-1)}\wedge n}\right] \\
&\leq \mathbb{E}\left[\frac{(X_n - a)^-}{b-a} : T_{2(k-1)} \leq n < T_{2k-1}\right] \\
&\leq \mathbb{E}\left[\frac{(X_n - a)^-}{b-a} : T_{2(k-1)-1} \leq n < T_{2(k-1)+1}\right] \\
&= \mathbb{E}\left[\frac{(X_n - a)^-}{b-a} : U_a^b(X;n) = k-1\right]
\end{aligned}$$

which proves the first inequality. Since $U_a^b(X,n) \leq n$ and takes values in non-negative integers, so that

$$\begin{aligned}
\mathbb{E}\left[U_a^b(X,n)\right] &= \sum_{k=1}^{\infty} k\mathbb{P}\left[U_a^b(X;n) = k\right] \\
&= \sum_{k=1}^{\infty} \mathbb{P}\left[U_a^b(X;n) \geq k\right] \\
&\leq \sum_{k=1}^{\infty} \mathbb{E}\left[\frac{(X_n - a)^-}{b-a} : U_a^b(X;n) = k-1\right] \\
&= \mathbb{E}\left[\frac{(X_n - a)^-}{b-a}\right].
\end{aligned}$$

2) If X is a sub-martingale, then $\mathbb{E}\left(X_{T_{2k-1}\wedge n} - X_{T_{2k}\wedge n}\right) \leq 0$, so that, by (11.4) we obtain

$$\begin{aligned}
\mathbb{P}\left[U_a^b(X;n) \geq k\right] &\leq \mathbb{E}\left[\frac{X_n - a}{b-a} : T_{2k-1} \leq n < T_{2k}\right] \\
&\leq \mathbb{E}\left[\frac{(X_n - a)^+}{b-a} : U_a^b(X;n) = k\right]
\end{aligned}$$

and therefore

$$\begin{aligned}\mathbb{E}\left[U_a^b(X,n)\right] &= \sum_{k=1}^{\infty}\mathbb{P}\left[U_a^b(X;n)\geq k\right]\\ &\leq \sum_{k=1}^{\infty}\mathbb{E}\left[\frac{(X_n-a)^+}{b-a} : U_a^b(X;n)=k\right]\\ &\leq \mathbb{E}\left[\frac{(X_n-a)^+}{b-a}\right]\end{aligned}$$

which completes the proof. ■

11.2 Martingale convergence theorem

Theorem 11.3 (The martingale convergence theorem, *J. L. Doob) 1) Suppose* $X=(X_n)_{n\geq 0}$ *is a super-martingale (or a sub-martingale), bounded in* $L^1(\Omega,\mathcal{F},\mathbb{P})$, *that is,* $\sup_n \mathbb{E}[|X_n|]<\infty$, *then* $X_\infty=\lim_{n\to\infty}X_n$ *exists almost surely and* $X_\infty\in L^1(\Omega)$.

2) If $Y=(Y_n)_{n\geq 0}$ *is non-negative super-martingale, and bounded in* L^1, *then* $Y_n\to Y_\infty$ *exists,* $Y_\infty\in L^1$, *and* $\mathbb{E}[Y_\infty|\mathcal{F}_n]\leq Y_n$ *for* $n\geq 0$.

3) If $M=(M_n)$ is uniformly integrable martingale, *that is,* $M=(M_n)_{n\geq 1}$ *is a martingale, and* $\{M_n : n=0,1,\cdots\}$ *is uniformly integrable, then* $M_\infty=\lim_{n\to\infty}M_n$ *exists almost surely,* $M_n\to M_\infty$ *in* $L^1(\Omega)$, *and* $M_n=\mathbb{E}[M_\infty|\mathcal{F}_n]$ *for every* n.

Proof. [*The proof is not examinable*] 1) For every pair of real numbers $a<b$, $U_a^b(X)=\lim_{n\to\infty}U_a^b(X;n)$ is the total number of up-crossings made by (X_n) through the interval (a,b). By MCT and Doob's crossing lemma we have

$$\begin{aligned}\mathbb{E}\left[U_a^b(X)\right] &= [\text{due to MCT}]\lim_{n\to\infty}\mathbb{E}\left[U_a^b(X;n)\right]\\ &\leq \sup_n \mathbb{E}\left[\frac{(X_n-a)^-}{b-a}\right]\\ &\leq \frac{|a|}{b-a}+\frac{1}{b-a}\sup_n\mathbb{E}\left[|X_n|\right]<\infty.\end{aligned}$$

That is, $U_a^b(X)$ is integrable, hence $U_a^b(X)$ is finite almost surely. Let

$$W_{(a,b)}=\{\text{liminf}_{n\to\infty}X_n<a,\ \text{limsup}_{n\to\infty}X_n>b\}$$

and

$$W=\bigcup\left\{W_{(a,b)} : a<b \text{ and } a,b \text{ are rationals }\right\}.$$

where the union runs through the countable set of rational pairs (a,b), $a<b$. Then $W_{(a,b)}\subset\left\{U_a^b(X)=\infty\right\}$, so that $\mathbb{P}\left[W_{(a,b)}\right]=0$. Hence $\mathbb{P}(W)=0$. However, if $\omega\notin W$, then $\liminf X_n(\omega)=\limsup X_n(\omega)$, so that $\lim_{n\to\infty}X_n(\omega)$ exists (or equals $\pm\infty$), denoted it by $X_\infty(\omega)$, and we set $X_\infty(\omega)=0$ for $\omega\in W$. Then $X_n\to X_\infty$ on W^c, so that $X_n\to X_\infty$ almost surely. According to Fatou's lemma

$$\mathbb{E}[|X_\infty|]=\mathbb{E}\left[\lim_{n\to\infty}|X_n|\right]\leq\liminf_{n\to\infty}\mathbb{E}[|X_n|]\leq\sup_n\mathbb{E}|X_n|<\infty$$

so that $X_\infty\in L^1(\Omega,\mathcal{F},\mathbb{P})$. We have thus proved the first part of the theorem.

2) Since Y is a non-negative bounded super-martingale, then by 1) $Y_n \to Y_\infty$ a.e. Since $\mathbb{E}[Y_m|\mathcal{F}_n] \le Y_n$ for $m \ge n$, letting $m \to \infty$, by Fatou's lemma (for conditional expectations),

$$\mathbb{E}[Y_\infty|\mathcal{F}_n] = \mathbb{E}\left[\lim_{m\to\infty} Y_m|\mathcal{F}_n\right] \le \liminf_{m\to\infty} \mathbb{E}[Y_m|\mathcal{F}_n] \le Y_n$$

the proof is therefore complete.

3) If $M = (M_n)$ is uniformly integrable martingale, then $\{M_n : n = 0, 1, \cdots\}$ is bounded, so that by 1), $M_n \to M_\infty$ almost surely, and hence $M_n \to M_\infty$ in L^1. Since for every $m > n$ we have $\mathbb{E}[M_m|\mathcal{F}_n] = M_n$, by letting $m \to \infty$ to obtain $M_n = \mathbb{E}[M_\infty|\mathcal{F}_n]$. ■

Recall that if $X = (X_n)$ is a SMartingale which is uniformly integrable, then $X_n \to X_\infty$ almost surely and in L^1. For the L^p-bounded martingale, we have the following

Theorem 11.4 *Suppose $X = (X_n)_{n\ge1}$ is a martingale which is bounded in L^p-space for some $p > 1$, that is, $\sup_n \mathbb{E}[|X_n|^p] < \infty$, then $(X_n)_{n\ge0}$ is uniformly integrable, and $X_n \to X_\infty$ almost surely, and in L^p-space. Moreover*

$$\mathbb{E}[|X_\infty|^p] = \sup_n \mathbb{E}[|X_n|^p].$$

Proof. [*The proof is not examinable.*] It is known that $\sup_n \mathbb{E}[|X_n|^p] < \infty$ for some $p > 1$ implies that (X_n) is uniformly integrable, so that $X_n \to X_\infty$ almost surely and in L^1. Let $g = \lim_{n\to\infty} \sup_{k\le n} |X_k|^p$. Applying Doob's L^p-inequality to the sub-martingale $|X_n|^p$ we have

$$\mathbb{E}\left[\sup_{k\le n} |X_k|^p\right] \le \left(\frac{p}{p-1}\right)^p \mathbb{E}[|X_n|^p] \le \left(\frac{p}{p-1}\right)^p \sup_n \mathbb{E}[|X_n|^p].$$

Thus, by MCT we conclude that

$$\mathbb{E}[g] = \lim_{n\to\infty} \mathbb{E}\left[\sup_{k\le n} |X_k|^p\right] \le \left(\frac{p}{p-1}\right)^p \sup_n \mathbb{E}[|X_n|^p] < \infty$$

that is g is integrable. Now $|X_n - X_\infty|^p \to 0$ almost surely, and $|X_n - X_\infty|^p \le 2^p g$ for all n, so by Lebesgue's dominated convergence theorem, we have

$$\mathbb{E}[|X_n - X_\infty|^p] \to 0$$

as $n \to \infty$. Since $|X_n|^p$ is a sub-martingale, so that $n \to \mathbb{E}[|X_n|^p]$ is increasing, and therefore

$$\mathbb{E}[|X_\infty|^p] = \lim_{n\to\infty} \mathbb{E}[|X_n|^p] = \sup_n \mathbb{E}[|X_n|^p].$$

Downward martingale convergence theorem

Let us now consider backward martingale convergence theorem.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, instead of a filtration, we are given a decreasing family of sub σ-algebras $(\mathcal{G}_n)_{n\geq 0}$, where $\mathcal{G}_{n+1} \subseteq \mathcal{G}_n$ for $n = 0, 1, 2, \cdots$, where the largest σ-algebra is the initial one $\mathcal{G}_0 \subset \mathcal{F}$. The final σ-algebra is $\mathcal{G}_\infty = \lim_{n\to\infty} \mathcal{G}_n = \bigcap_{j=0}^{\infty} \mathcal{G}_j$.

We may define martingales, sub-martingales and super-martingales with respect to the decreasing flow $(\mathcal{G}_n)$. Namely, a $(\mathcal{G}_n)$-adapted and integrable random sequence $X = (X_n)_{n\geq 0}$ is a martingale (resp. super-martingale, and resp. sub-martingale) if $\mathbb{E}[X_n|\mathcal{G}_{n+1}] = X_{n+1}$ (resp. $\mathbb{E}[X_n|\mathcal{G}_{n+1}] \leq X_{n+1}$, and resp. $\mathbb{E}[X_n|\mathcal{G}_{n+1}] \geq X_{n+1}$).

If we set $\mathcal{F}_n = \mathcal{G}_{-n}$ where $n = \cdots, -2, -1, 0$ (with the natural order in $\mathbb{Z}_-$), then $(\mathcal{F}_n)$ (where $n = \cdots, -2, -1, 0$) is a filtration, i.e. an increasing flow of σ-algebras. Then $M_n = X_{-n}$ (where $n = \cdots, -2, -1, 0$) is martingale (resp. super-martingale, resp. sub-martingale) if $\mathbb{E}[M_n|\mathcal{F}_{n-1}] = M_{n-1}$ (resp. $\mathbb{E}[M_n|\mathcal{F}_{n-1}] \leq M_{n-1}$, resp. $\mathbb{E}[M_n|\mathcal{F}_{n-1}] \geq M_{n-1}$) for $n = \cdots, -2, -1, 0$. The following technical lemma allows us apply the results we have established in the previous sections to martingales with respect to a decreasing flow of σalgebras, which follows directly from the definition.

Lemma 11.5 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space together with a decreasing family $(\mathcal{G}_n)_{n\geq 0}$ of sub σ-algebras of $\mathcal{F}$. Let $X = (X_n)_{n\geq 0}$, where $X_n \in L^1(\Omega, \mathcal{G}_n, \mathbb{P})$ for $n = 0, 1, 2, \cdots$. Then, X is martingale (resp. super-martingale, resp. sub-martingale), if and only if for every $N = 1, 2, \cdots$, the* time-reversed *random sequence $Y_n = X_{N-n}$ (where $n = 0, \cdots, N$) is a martingale (resp. super-martingale, resp. sub-martingale) up to time N (with terminal value X_0), with respect to the filtration $\mathcal{G}_{N-n}$.*

As a sample of applications of the previous lemma, we prove the following very useful convergence result.

Theorem 11.6 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space together with a decreasing family $(\mathcal{G}_n)_{n\geq 0}$ of sub σ-algebras of $\mathcal{F}$. If $X = (X_n)_{n\geq 0}$ is a super-martingale with respect to $(\mathcal{G}_n)$, then $X_\infty = \lim_{n\to\infty} X_n$ exists almost surely. If in addition $\lim_{n\to\infty} \mathbb{E}[X_n] < \infty$ then $\{X_n : n \geq 0\}$ is uniformly integrable, and $X_n \to X_\infty$ in $L^1(\Omega)$.*

Proof. [*The proof is not examinable.*] For every $N = 1, 2, \cdots$, the time-reversed sequence $\{X_N, X_{N-1}, \cdots, X_0\}$ is a super-martingale (up to time N) with respect to $\mathcal{G}_{N-n}$, its up-crossing number through $[a, b]$ where $a < b$ is denoted by $U_a^b(X, -N)$. The label $-$ is used to indicate the reversed up-crossing, rather than $U_a^b(X, N)$ which is the up-crossing of $\{X_0, X_1, \cdots, X_N\}$, but they are equally useful in determining the convergence. Let $U_a^b(X) = \lim_{N\to\infty} U_a^b(X, -N)$ which represents the number of up-crossings for the time-reversed sequence $\{\cdots, X_N, X_{N-1}, \cdots, X_0\}$. According to Doob's up-crossing lemma, for every N,

$$\mathbb{E}\left[U_a^b(X; -N)\right] \leq \mathbb{E}\left[\frac{(X_0 - a)^-}{b - a}\right].$$

The right-hand side is independent of N, so by applying the Monotone Convergence Theorem, we have

$$\mathbb{E}\left[U_a^b(X)\right] \leq \mathbb{E}\left[\frac{(X_0 - a)^-}{b - a}\right].$$

Therefore $U_a^b(X)$ is integrable, so that $U_a^b(X) < \infty$ almost everywhere. A similar argument as the proof of the Martingale Convergence Theorem may apply to conclude that $X_\infty = \lim_{n\to\infty} X_n$ exists almost everywhere, and X_∞ is $\bigcap_{j=1}^{\infty} \mathcal{G}_j$-measurable.

Since $n \to \mathbb{E}[X_n]$ is increasing (note that not decreasing, as it is a time-reversed super-martingale), so that $\sup_n \mathbb{E}[X_n] = \lim_{n\to\infty} \mathbb{E}[X_n]$. Suppose that $\lim_{n\to\infty} \mathbb{E}[X_n] < \infty$. Then $\sup_n \mathbb{E}[X_n] < \infty$. Since X_0 is integrable, $\xi_n = \mathbb{E}[X_0|\mathcal{G}_n]$ is uniformly integrable (time-reversed) martingale, and $Q_n = X_n - \xi_n$ is (time-reversed) super-martingale. Since

$$Q_n = \mathbb{E}[Q_n|\mathcal{G}_n] = \mathbb{E}[X_n - X_0|\mathcal{G}_n] \geq 0$$

which implies that Q_n is non-negative, and $X_n = Q_n + \xi_n$. Therefore, to show that X is uniformly integrable, we only need to show that $Q = (Q_n)$ is uniformly integrable. Thus, without losing generality, we may assume that $X = (X_n)$ is a non-negative (time-reversed) super-martingale, and $\sup_n \mathbb{E}[X_n] = \lim_{n\to\infty} \mathbb{E}[X_n] < \infty$.

According to the time-reversed super-martingale property, for any $n > m \geq 0$ and $L > 0$, since $\{X_n \leq L\} \in \mathcal{G}_n$ we have

$$\begin{aligned}\mathbb{E}[X_n : X_n > L] &= \mathbb{E}[X_n] - \mathbb{E}[X_n : X_n \leq L] \leq \mathbb{E}X_n - \mathbb{E}[X_m : X_n \leq L] \\ &\leq \mathbb{E}[X_n] - \mathbb{E}[X_m] + \mathbb{E}[X_m : X_n > L].\end{aligned}$$

Since $\lim_{n\uparrow\infty} \mathbb{E}[X_n]$ exists, so for every $\varepsilon > 0$, there is N_1 such that $0 \leq \mathbb{E}[X_n] - \mathbb{E}[X_m] < \frac{\varepsilon}{2}$ for all $n, m \geq N_1$. Since the finite family of integrable random variables$\{X_0, \cdots, X_{N_1}\}$ is uniformly integrable, so there is $\delta > 0$ such that $\mathbb{E}[X_m : A] < \varepsilon/2$ for any A with $\mathbb{P}(A) < \delta$, for all $m \leq N_1$. On the other hand, using Markov inequality $\mathbb{P}[X_n > L] \leq \frac{\sup_n \mathbb{E}X_n}{L}$. Choose $L_0 = \frac{\sup_n \mathbb{E}X_n}{\delta}$. Then $\mathbb{P}[X_n > L] < \delta$ for all $L \geq L_0$ and for all n. Therefore $\mathbb{E}[X_m : X_n > L] < \frac{\varepsilon}{2}$ for all $m \leq N_1$ and $L \geq L_1$, and

$$\mathbb{E}[X_n : X_n > L] \leq \mathbb{E}X_n - \mathbb{E}X_{N_1} + \mathbb{E}[X_{N_1} : X_n > L] < \varepsilon$$

for all $L \geq L_0$ and $n \geq N_1$. Putting all these estimates together we deduce that

$$\mathbb{E}[X_n : X_n > L] < \varepsilon$$

for all n and $L \geq L_0$, which proves that (X_n) is uniformly integrable. Hence $X_n \to X_\infty$ in $L^1(\Omega)$ as well. ∎