FIN 508- MARTINGALE THROUGH MEASURE THEORY
LECTURE 8

UNIFORM INTEGRABILITY

1. Definition of uniform integrability. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The concept of uniform integrability for a family of integrable functions is used to handle the convergence in $L^1(\Omega)$. In spirit, it is very close to that of uniform convergence, uniform continuity etc. that you have learned in the analysis course. If f is integrable, then f is finite almost everywhere. Hence $|f|1_{\{|f|<N\}} \uparrow |f|$ almost everywhere as $N \uparrow \infty$, thus by the Monotone Convergence Theorem $\int_\Omega |f|1_{\{|f|<N\}}d\mathbb{P} \uparrow \int_\Omega |f|d\mathbb{P}$, so that $\lim_{N\to\infty} \int_{\{|f|\geq N\}} |f|\mathrm{d}\mathbb{P} = 0$.

Definition 8.1 *Let $\mathcal{A}$ be a family of integrable functions on $(\Omega, \mathcal{F}, \mu)$. $\mathcal{A}$ is uniformly integrable if*

$$\lim_{N\to\infty} \sup_{\xi\in\mathcal{A}} \int_{\{|\xi|\geq N\}} |\xi| d\mathbb{P} = 0 \ .$$

That is, $\mathbb{E}\left[|\xi| : |\xi| \geq N\right]$ tends to zero uniformly on $\mathcal{A}$ as $N \to \infty$.

2. Some simple properties.

2.1) Any finite family of integrable random variables is uniformly integrable.

2.2) Suppose $\mathcal{A} \subset L^1(\Omega)$ and there is $\eta \in L^1(\Omega)$ such that $|\xi| \leq \eta$ for every $\xi \in \mathcal{A}$, then $\mathcal{A}$ is uniformly integrable.

2.3) $\mathcal{A} \subset L^p(\Omega)$ such that $\sup_{\xi\in\mathcal{A}} \int_\Omega |\xi|^p d\mathbb{P} < \infty$ for some $p > 1$ [which is equivalent to that $\mathcal{A}$ is bounded in $L^p(\Omega)$], then $\mathcal{A}$ is uniformly integrable. In fact,

$$\begin{aligned} \sup_{\xi\in\mathcal{A}} \int_{\{|\xi|\geq N\}} |\xi| \mathrm{d}\mu &\leq \sup_{\xi\in\mathcal{A}} \int_{\{|\xi|\geq N\}} \frac{1}{N^{p-1}} |\xi|^p \mathrm{d}\mu \\ &\leq \frac{1}{N^{p-1}} \sup_{\xi\in\mathcal{A}} \mathbb{E}\left[|\xi|^p\right] \to 0. \end{aligned}$$

Theorem 8.2 *Let $\mathcal{A} \subset L^1(\Omega)$. Then $\mathcal{A}$ is uniformly integrable if and only if*

(a) $\mathcal{A}$ is a bounded subset of $L^1(\Omega)$, that is, $\sup_{\xi\in\mathcal{A}} \mathbb{E}\left[|\xi|\right] < \infty$.

(b) For any $\varepsilon > 0$ there is a $\delta > 0$ such that $\sup_{\xi\in\mathcal{A}} \mathbb{E}\left[|\xi| : E\right] \leq \varepsilon$ whenever $E \in \mathcal{F}$ with $\mu(E) \leq \delta$.

Proof. Suppose $\mathcal{A}$ is uniformly integrable. For any $E \in \mathcal{F}$ and $N > 0$

$$\begin{aligned} \int_E |\xi| \mathrm{d}\mathbb{P} &= \int_{E\cap\{|\xi|<N\}} |\xi| \mathrm{d}\mathbb{P} + \int_{E\cap\{|\xi|\geq N\}} |\xi| \mathrm{d}\mathbb{P} \\ &\leq N + \int_{\{|\xi|\geq N\}} |\xi| \mathrm{d}\mathbb{P} \ . \end{aligned}$$

Given $\varepsilon > 0$, choose $N > 0$ such that $\sup_{\xi\in\mathcal{A}} \mathbb{E}\left[|\xi| : |\xi| \geq N\right] \leq \varepsilon/2$. Then $\sup_{\xi\in\mathcal{A}} \mathbb{E}\left[|\xi| : E\right] \leq N + \varepsilon/2$ for any $E \in \mathcal{F}$. Thus$\delta = \varepsilon/(4N)$ will do.

Conversely, suppose 1) and 2) are satisfied. Let $\beta = \sup_{\xi\in\mathcal{A}} \mathbb{E}\left[|\xi|\right]$. Then, by the Markov inequality, $\mathbb{P}\left\{|\xi| \geq N\right\} \leq \beta/N$ for any $N > 0$. For any $\varepsilon > 0$, there is a $\delta > 0$ such that the inequality in 2) holds. Let $N = \beta/\delta$. Then $\mathbb{P}\left\{|\xi| \geq N\right\} \leq \delta$ so that $\mathbb{E}\left[|\xi| : |\xi| \geq N\right] \leq \varepsilon$ for any $\xi \in \mathcal{A}$. ■

Corollary 8.3 *Suppose $\mathcal{A} \subset L^1(\Omega)$ and $\eta \in L^1(\Omega)$ such that $\mathbb{E}\left[1_D|\xi|\right] \leq \mathbb{E}\left[1_D|\eta|\right]$ for any $D \in \mathcal{F}$ and $\xi \in \mathcal{A}$. Then $\mathcal{A}$ is uniformly integrable.*

3. L^1-convergence and uniform integrability. The following theorem demonstrates the importance of uniform integrability.

Theorem 8.4 *Let f_n be a sequence of integrable functions on $(\Omega, \mathcal{F}, \mathbb{P})$. Then $f_n \to f$ in $L^1(\Omega)$ as $n \to \infty$:*

$$||f_n - f||_{L^1(\Omega)} = \mathbb{E}\left[|f_n - f|\right] \to 0 \quad as \quad n \to \infty \ ,$$

if and only if $\{f_n\}$ is uniformly integrable and $f_n \to f$ in measure as $n \to \infty$.

Proof. *Necessity.* For any $\varepsilon > 0$ there is a natural number m such that $||f_n - f||_{L^1(\Omega)} < \varepsilon/2$ for all $n > m$. Therefore, for every measurable subset E,

$$\sup_n \int_E |f_n| \mathrm{d}\mathbb{P} \leq \int_E |f| \mathrm{d}\mathbb{P} + \sup_{k \leq m} \int_E |f_k| \mathrm{d}\mathbb{P} + \frac{\varepsilon}{2} \ .$$

In particular

$$\sup_n \mathbb{E}\left[|f_n|\right] \leq \mathbb{E}\left[|f|\right] + \sup_{k \leq m} \mathbb{E}\left[|f_k|\right] + \frac{\varepsilon}{2}$$

i.e. $\{f_n : n \geq 1\}$ is bounded in $L^1(\Omega)$. Moreover, since $f, f_1, \cdots, f_m$ belong to L^1, so that there is $\delta > 0$ such that, if $\mathbb{P}(E) \leq \delta$, then

$$\int_E |f| \mathrm{d}\mathbb{P} + \sum_{k=1}^{m} \int_E |f_k| \mathrm{d}\mathbb{P} \leq \frac{\varepsilon}{2}.$$

Therefore $\sup_n \int_E |f_n| \mathrm{d}\mathbb{P} \leq \varepsilon$ as long as $\mu(E) \leq \delta$.

Sufficiency. By Fatou's lemma $\int_\Omega |f| \mathrm{d}\mathbb{P} \leq \sup_n \int_\Omega |f_n| \mathrm{d}\mathbb{P}$, so that $f \in L^1(\Omega)$. Therefore $\{f_n - f : n \geq 1\}$ is uniformly integrable, thus, by Theorem 8.2, for any $\varepsilon > 0$ there is $\delta > 0$ such that $\int_E |f_n - f| \mathrm{d}\mathbb{P} < \varepsilon$ for any $E \in \mathcal{F}$ satisfying that $\mathbb{P}(E) \leq \delta$. Since $f_n \to f$ in probability, there is an $N > 0$ such that $\mathbb{P}\left(|X_n - X| \geq \varepsilon\right) \leq \delta$ for any $n \geq N$. Therefore

$$\begin{aligned}\int_\Omega |f_n - f| \mathrm{d}\mathbb{P} &\leq \int_{\{|X_n - X| \geq \varepsilon\}} |f_n - f| \mathrm{d}\mathbb{P} + \varepsilon \mathbb{P}\left\{f_n - f| < \varepsilon\right\} \\ &\leq \varepsilon + \varepsilon \mathbb{P}\left\{|f_n - f| < \varepsilon\right\} \\ &\leq 2\varepsilon \ .\end{aligned}$$

for $n \geq N$. By definition, $f_n \to f$ in $L^1(\Omega)$. ∎