FIN 508- MARTINGALE THROUGH MEASURE THEORY
LECTURE 5

PRODUCT MEASURES AND FUBINI'S THEOREM

1. Product of several σ-algebras. Let A and B be two sets. Then $A \times B$ (the product set) is the set of all ordered pairs (x, y) where $x \in A$ and $y \in B$. Let Ω_1 and Ω_2 be two spaces. Then $\Omega_1 \times \Omega_2$ is also called the Cartesian product space. Suppose $\mathcal{F}_1$ and $\mathcal{F}_2$ are algebras on spaces Ω_1 and Ω_2 respectively, then $\mathcal{F}_1 \times \mathcal{F}_2$ is in general not an algebra, but the collection of all finite unions $\bigcup_{j=1}^{k} A_j \times B_j$ (where $A_j \in \mathcal{F}_1$ and $B_j \in \mathcal{F}_2$ and k is a positive integer) is an algebra. If $\mathcal{F}_i$ are σ-algebras, $\mathcal{F}_1 \times \mathcal{F}_2$ is in general not a σ-algebra, and we define $\mathcal{F}_1 \otimes \mathcal{F}_2$ to be the smallest σ-algebra containing $\mathcal{F}_1 \times \mathcal{F}_2$, that is, $\mathcal{F}_1 \otimes \mathcal{F}_2 = \sigma\{\mathcal{F}_1 \times \mathcal{F}_2\}$. The construction may be extended to the product space of finite many spaces. More precisely, if $(\Omega_i, \mathcal{F}_i)$ $(i = 1, \cdots, n)$ are measurable spaces, then

$$\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_n = \sigma\{A_1 \times \cdots \times A_n : A_i \in \mathcal{F}_i\}$$

and $(\Omega_1 \otimes \cdots \otimes \Omega_n, \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_n)$ is called the product measurable space of $(\Omega_i, \mathcal{F}_i)$.

Exercise 5.1 *1) Suppose S_i $(i = 1, \cdots, n)$ are topological spaces with countable basis, so that the product space $S_1 \times \cdots \times S_n$ carries the product topology. Show that*

$$\mathcal{B}(S_1 \times \cdots \times S_n) = \mathcal{B}(S_1) \otimes \cdots \otimes \mathcal{B}(S_n).$$

2) If $(\Omega_i, \mathcal{F}_i)$ $(i = 1, 2, \cdots)$ are measurable spaces, then

$$\begin{aligned} \mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \mathcal{F}_3 &= \mathcal{F}_1 \otimes (\mathcal{F}_2 \otimes \mathcal{F}_3) \\ &= (\mathcal{F}_1 \otimes \mathcal{F}_2) \otimes \mathcal{F}_3. \end{aligned}$$

2. Product σ-algebra of countable many σ-algebras. Let us now consider a sequence of measurable spaces $(\Omega_i, \mathcal{F}_i)$ $(i = 1, 2, \cdots)$. The Cartesain product $\prod_{i=1}^{\infty} \Omega_i$ is the space consisting of all sequences $(x_1, \cdots, x_i, \cdots)$ where $x_i \in \Omega_i$ for $i = 1, 2, \cdots$, and define $\prod_{i=1}^{\infty} \otimes \mathcal{F}_i$ to be the smallest σ-algebra containing all $\prod_{i=1}^{\infty} A_i$ where $A_i \in \mathcal{F}_i$ for all i and $A_i = \Omega_i$ except *for finite many* $i \in \mathbb{N}$. $(\prod_{i=1}^{\infty} \Omega_i, \prod_{i=1}^{\infty} \otimes \mathcal{F}_i)$ is called the product measurable space of $(\Omega_i, \mathcal{F}_i)$, $i = 1, 2, \cdots$.

3. Measurable sections. Now let us come to the construction of product measures on product spaces. We need the following elementary fact.

Lemma 5.2 *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are algebras on Ω_1 and Ω_2 respectively, then the collection $\mathcal{A}(\mathcal{F}_1, \mathcal{F}_2)$ of all finite disjoint unions $\bigcup_{i=1}^k A_i \times B_i$ for some $k \in \mathbb{N}$, where $A_i \in \mathcal{F}_1$, $B_i \in \mathcal{F}_2$ and all products $A_i \times B_i$ are disjoint, is an algebra on $\Omega_1 \times \Omega_2$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are σ-algebras, then $\mathcal{F}_1 \otimes \mathcal{F}_2 = \sigma\{\mathcal{A}(\mathcal{F}_1, \mathcal{F}_2)\}$.*

Lemma 5.3 *Let $(\Omega_i, \mathcal{F}_i)$ $(i = 1, 2)$ be measurable spaces. 1) If $A \in \mathcal{F}_1 \otimes \mathcal{F}_2$, then for each $x_1 \in \Omega_1$ the section*

$$A_{x_1} = \{x_2 \in \Omega_2 : (x_1, x_2) \in A\}$$

is measurable, i.e. $A_{x_1} \in \mathcal{F}_2$. Similarly

$$A^{x_2} = \{x_1 \in \Omega_1 : (x_1, x_2) \in A\}$$

belongs to $\mathcal{F}_1$.

2) Suppose f is measurable on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$, then for each $x_1 \in \Omega_1$, the function $f_{x_1}(x_2) = f(x_1, x_2)$ is $\mathcal{F}_2$-measurable.

Proof. Proof of 1). Let $\mathcal{E}$ be the collection of all $E \subseteq \Omega_1 \times \Omega_2$ such that its x_1-section is measurable. Then $\mathcal{E}$ is a σ-algebra containing all $A \times B$ where $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$. Therefore $\mathcal{F}_1 \otimes \mathcal{F}_2 \subset \mathcal{E}$ which proves 1). To show 2), we notice that

$$\{x_2 : f_{x_1}(x_2) > a\} = \{x_2 : f(x_1, x_2) > a\}$$

which is the x_1-section of $\{f > a\}$ (which is $\mathcal{F}_1 \otimes \mathcal{F}_2$-measurable), so its x_1-section is $\mathcal{F}_2$-measurable. Therefore f_{x_1} is $\mathcal{F}_2$-measurable. ∎

In particular, if S_i are topological spaces with Borel σ-algebras, and if f is Borel measurable on $S_1 \times S_2$ with the product topology, then its section f_{x_1} (for each $x_1 \in S_1$) is Borel measurable on S_2.

4. Product measure of two measures. The following is the main technical fact in the construction of product measures.

Lemma 5.4 *Let $(\Omega_i, \mathcal{F}_i, \mu_i)$ $(i = 1, 2)$ be two finite measure spaces. Then for any $A \in \mathcal{F}_1 \otimes \mathcal{F}_2$, $x_1 \to \mu_2(A_{x_1})$ (resp. $x_2 \to \mu_1(A^{x_2})$) is measurable on $(\Omega_1, \mathcal{F}_1)$ (resp. $(\Omega_2, \mathcal{F}_2)$) and*

$$\int_{\Omega_1} \mu_2(A_{x_1})\mu_1(dx_1) = \int_{\Omega_2} \mu_1(A^{x_2})\mu_2(dx_2) \tag{5.1}$$

the common value is denoted by $\mu_1 \times \mu_2(A)$, so that $\mu_1 \times \mu_2$ is defined on $\mathcal{F}_1 \otimes \mathcal{F}_2$.

Proof. Let $\mathscr{L}$ denote the collection of all subsets $A \in \mathcal{F}_1 \otimes \mathcal{F}_2$ such that both functions $\mu_2(A_{x_1})$ and $\mu_1(A^{x_2})$ are measurable and (5.1) holds. By definition, $\mathcal{F}_1 \times \mathcal{F}_2 \subset \mathscr{L}$, and by linearity of integration, we can see that $\mathscr{L}$ is a ring. On the other hand, by using MCT, we can show that $\mathscr{L}$ is a monotone class. Therefore $\mathscr{L}$ must be a σ-algebra, so that $\mathscr{L} = \mathcal{F}_1 \otimes \mathcal{F}_2$. ∎

Theorem 5.5 *Let $(\Omega_i, \mathcal{F}_i, \mu_i)$ $(i = 1, 2)$ be two σ-finite measure spaces. Choose a sequence $G_n = A_n \times B_n$, where $A_n \uparrow \Omega_1$, $A_n \in \mathcal{F}_1$, $\mu_1(A_n) < \infty$, and similarly, $B_n \uparrow \Omega_2$, $B_n \in \mathcal{F}_2$, $\mu_2(B_n) < \infty$, for every n. If $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$ then define*

$$m(E) = \lim_{n\to\infty} \mu_1 \times \mu_2(E \cap G_n)$$

where $\mu_1 \times \mu_2(E \cap G_n)$ is defined in Lemma 5.4. Then m is the unique σ-finite measure on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$, such that

$$m(A \times B) = \mu_1(A)\mu_2(B) \quad \forall A \in \mathcal{F}_1,\ B \in \mathcal{F}_2. \tag{5.2}$$

which will be denoted by $\mu_1 \times \mu_2$, called the product measure of μ_1 and μ_2.

Proof. Uniqueness follows from Lemma 2.3. Given a sequence $\{G_n\}$ satisfying the conditions in the theorem. Since $\mu_1 \times \mu_2(E \cap G_n)$ is non-negative and increasing, so that m is well defined on $\mathcal{F}_1 \otimes \mathcal{F}_2$. Clearly $m(\emptyset) = 0$, so we need to show that m is countably additive. We prove this in two steps.

Note that $\mu_1(\cdot \cap A_n)$ and $\mu_2(\cdot \cap B_n)$ are finite measures, so that $\mu_1 \times \mu_2(E \cap G_n)$ is well-defined via (5.1), and is non-negative, increasing in n. We want to show that m is countably additive. Suppose $E_k \in \mathcal{F}_1 \otimes \mathcal{F}_2$ are disjoint sequence, and $E = \cup_{k=1}^{\infty} E_k$. Then, for every n

$$\begin{aligned}
m(E \cap G_n) &= \int_{\Omega_2} \mu_1((E \cap G_n)^{x_2})\mu_2(dx_2) = \int_{\Omega_2} \mu_1(\cup_k (E_k \cap G_n)^{x_2})\mu_2(dx_2) \\
&= \int_{\Omega_2} \sum_k \mu_1((E_k \cap G_n)^{x_2})\mu_2(dx_2) = \sum_k \int_{\Omega_2} \mu_1((E_k \cap G_n)^{x_2})\mu_2(dx_2) \\
&= \sum_k m(E_k \cap G_n).
\end{aligned}$$

where the fourth equality follows from MCT (series version). It follows that

$$m(E \cap G_n) \le \sum_k m(E_k)$$

so that, by letting $n \to \infty$ we obtain $m(E) \le \sum_k m(E_k)$. On the other hand, for every N,

$$m(E \cap G_n) \ge \sum_{k=1}^{N} m(E_k \cap G_n).$$

Letting $n \to \infty$ we have $m(E) \geq \sum_{k=1}^{N} m(E_k)$, so that we also have $m(E) \geq \sum_k m(E_k)$. Therefore $m(E) = \sum_k m(E_k)$ which completes the proof. ■

5. Product measure of finite many σ-finite measures. Obviously, the same approach is applied to finite many σ-finite measure spaces, and we have

Theorem 5.6 *Suppose $(\Omega_i, \mathcal{F}_i, \mu_i)$ $(i = 1, \cdots, n)$ are σ-finite measure spaces, then there is a unique σ-finite measure $\mu_1 \times \cdots \times \mu_n$ called the product measure on $(\Omega_1 \times \cdots \times \Omega_n, \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_n)$ such that*

$$\mu_1 \times \cdots \times \mu_n(A_1 \times \cdots \times A_n) = \mu_1(A_1) \cdots \mu_n(A_n) \quad \forall A_i \in \mathcal{F}_i.$$

6. Product probability measure of countable many probability measures. However, there is obstruction for constructing product measures on the product space of countably many measure spaces, one can not, in general, use $\prod_{i=1}^{\infty} \mu_i(A_i)$ to define the measure of $\prod_{i=1}^{\infty} A_i$ even if $A_i = \Omega_i$ except finite many i. This approach on the other hand works for probability spaces $(\Omega_i, \mathcal{F}_i, \mu_i)$ as in this case $\prod_{i=1}^{\infty} \mu_i(A_i)$ for $\prod_{i=1}^{\infty} A_i$, where $A_i = \Omega_i$ except finite many i, becomes a finite product as $\mu_i(\Omega_i) = 1$ for sufficient large i.

Theorem 5.7 *Suppose $(\Omega_i, \mathcal{F}_i, \mu_i)$ $(i = 1, 2, \cdots)$ are* probability *spaces, then there is a probability measure $\prod_{i=1}^{\infty} \mu_i$ (called the product probability measure) on $(\prod_{i=1}^{\infty} \Omega_i, \prod_{i=1}^{\infty} \otimes \mathcal{F}_i)$ such that*

$$\prod_{i=1}^{\infty} \mu_i(A_1 \times \cdots \times A_k \times \cdots) = \prod_{i=1}^{\infty} \mu_i(A_i).$$

for any $A_i \in \mathcal{F}_i$ for all i and $A_i = \Omega_i$ except for finite many i.

Proof. [*The proof is not examinable*] Let $\mathscr{R}$ denote the ring of all subsets $E \subset \prod_{i=1}^{\infty} \Omega_i$ which has the following form:

$$E = \bigcup_{j=1}^{n} A_j, \text{ where } A_j = A_1^{(j)} \times \cdots \times A_k^{(j)} \times \cdots$$

$A_k^{(j)} \in \mathcal{F}_k$ for $j = 1, \cdots, n$, and for every j, there is k_j, such that $A_k^{(j)} = \Omega_k$ for every $k > k_j$, for some $n \in \mathbb{N}$. If $E \in \mathscr{R}$ then we may choose a decomposition above such that A_j (for some n, $j = 1, \cdots, n$) are disjoint, and define

$$m(E) = \sum_{j=1}^{n} m(A_j) \text{ where } m(A_j) = \mu_1(A_1^{(j)}) \cdots \mu_k(A_k^{(j)}) \cdots$$

each $m(A_j)$ is in fact a finite product as all μ_k are probability measures. To see why m is well defined and is in fact a measure on $\mathscr{R}$, we make the following crucial observation. If $E_1, \cdots, E_N \in \mathscr{R}$, then, there is a common K, such that for all $n = 1, \cdots, N$ each $E_n = A^{(n)} \times \Omega_{K+1} \times \cdots$ for some $A^{(n)} \in \prod_{k=1}^{K} \mathcal{F}_k$, and therefore

$$E \equiv \bigcup_{n=1}^{N} E_n = A \times \Omega_{K+1} \times \cdots$$

for some $A \in \prod_{k=1}^{K} \mathcal{F}_k$. Since μ_k are probability measures, so by definition

$$m(E_n) = \mu_1 \times \cdots \times \mu_K(A^{(n)})$$

(the identity is no longer ensured if there are infinite many μ_k with total mass $\mu_k(\Omega_k) \neq 1$). Since $\mu_1 \times \cdots \times \mu_K$ is a measure, so that, if E_n $(n = 1, \cdots, N)$ are disjoint, then

$$m(E) = \mu_1 \times \cdots \times \mu_K(A) = \sum_{n=1}^{N} \mu_1 \times \cdots \times \mu_K(A^{(n)}) = \sum_{n=1}^{N} m(E_n)$$

which shows that m is well defined on the ring $\mathscr{R}$ and m is finitely additive. Next, the standard machinery may be applied to construct the product probability $\prod_{i=1}^{\infty} \mu_i$. Firstly, define outer measure

$$m^*(E) = \inf \left\{ \sum_{n=1}^{\infty} m(E_n) : \text{ where } E_n \in \mathscr{R} \text{ such that } \bigcup_{n=1}^{\infty} E_n \supset E \right\}$$

for every sunset $E \subset \prod_{i=1}^{\infty} \Omega_i$. Let $\mathcal{M}$ denote the σ-algebra of all m^*-measurable subsets. Then m^* is a measure on $\mathcal{M}$ (by the Carathodory extension theorem). Since $\mathscr{R}$ is a ring and m is finitely additive, we thus must have $\mathscr{R} \subset \mathcal{M}$. Since $\prod_{i=1}^{\infty} \otimes \mathcal{F}_i = \sigma(\mathscr{R}) \subset \mathcal{M}$, so that m^* restricted on $\prod_{i=1}^{\infty} \otimes \mathcal{F}_i$ is a probability measure. The construction is complete. ■

7. Fubini's theorem. Let us now turn to the Fubini theorem.

Let $(\Omega_i, \mathcal{F}_i, \mu_i)$ $(i = 1, 2)$ be two σ-finite measure spaces. Suppose $f : \Omega_1 \times \Omega_2 \to (-\infty, \infty)$ is a measurable function, such that for almost all $x_1 \in \Omega_1$, f_{x_1} is integrable on $(\Omega_2, \mathcal{F}_2, \mu_2)$. Hence, there is a set $N_1 \in \Omega_1$ with $\mu_1(N_1) = 0$, and for any $x_1 \in \Omega_1 \backslash N_1$, $f_{x_1} \in L^1(\Omega_2, \mathcal{F}_2, \mu_2)$, so that we can define

$$h(x_1) = \int_{\Omega_2} f_{x_1}(x_2) \mu_2(dx_2) \quad \text{if } x_1 \in \Omega_1 \setminus N_1$$

otherwise $h(x_1) = 0$. If there is a $\tilde{h} \in L^1(\Omega_1, \mathcal{F}_1, \mu_1)$, such that $\tilde{h} = h$ almost surely w.r.t. μ_1, then we can form an integral

$$I_{1,2}(f) = \int_{\Omega_1} \tilde{h}(x_1) \mu_1(dx_1).$$

One can show that, if $I_{1,2}(f)$ exists (i.e. there is some N_1 and $\tilde{h}$ satisfying the above conditions), then $I_{1,2}(f)$ does not depend on N_1 and $\tilde{h}$, therefore $I_{1,2}(f)$ is called an iterated integral of f over $\Omega_1 \times \Omega_2$, denoted by

$$\int_{\Omega_1} \left(\int_{\Omega_2} f(x_1, x_2) \mu_2(dx_2) \right) \mu_1(dx_1).$$

Similarly we define the iterated integral

$$\int_{\Omega_2} \left(\int_{\Omega_1} f(x_1, x_2) \mu_1(dx_1) \right) \mu_2(dx_2).$$

Theorem 5.8 *(*Fubini's theorem*) Let μ_j be σ-finite measure on $(\Omega_j, \mathcal{F}_j)$, where $j = 1, 2$. Suppose $f : \Omega_1 \times \Omega_2 \to (-\infty, \infty)$ is a measurable function on the product measure space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$.*

1) If $f \in L^1(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \times \mu_2)$, then both iterated integrals exist and equal to the integral $\int_{\Omega_1 \times \Omega_2} f d(\mu_1 \times \mu_2)$.

2) Conversely, if one of the iterated integral of $|f|$ is finite, then $f \in L^1(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \times \mu_2)$.

Proof. By Theorem 5.5 and the definition of the product measure $\mu_1 \times \mu_2$, for every $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$ we have

$$\int_{\Omega_1 \times \Omega_2} 1_E d\mu_1 \times \mu_2 = \int_{\Omega_2} \left[\int_{\Omega_1} 1_E d\mu_1 \right] d\mu_2 = \int_{\Omega_1} \left[\int_{\Omega_2} 1_E d\mu_2 \right] d\mu_1$$

which yields that Fubini's theorem holds for every non-negative simple measurable function.

Suppose f is non-negative and $\mathcal{F}_1 \otimes \mathcal{F}_2$-measurable, then we can choose a sequence of non-negative, measurable simple functions $\varphi_n : \Omega_1 \times \Omega_2 \to [0, \infty)$ such that $\varphi_n \uparrow f$. By MCT we have

$$\begin{aligned}\int_{\Omega_1 \times \Omega_2} f d\mu_1 \times \mu_2 &= \lim_{n \to \infty} \int_{\Omega_1 \times \Omega_2} \varphi_n d\mu_1 \times d\mu_2 \\ &= \lim_{n \to \infty} \int_{\Omega_2} \left[\int_{\Omega_1} \varphi_n d\mu_1 \right] d\mu_2 = \lim_{n \to \infty} \int_{\Omega_2} \Phi_n d\mu_2\end{aligned}$$

where

$$\Phi_n = \int_{\Omega_1} \varphi_n d\mu_1$$

which are non-negative, $\mathcal{F}_2$-measurable and $\Phi_n \uparrow$, thus by MCT applying to $\{\Phi_n\}$ on $(\Omega_2, \mathcal{F}_2, \mu_2)$ to obtain

$$\lim_{n \to \infty} \int_{\Omega_2} \Phi_n d\mu_2 = \int_{\Omega_2} \lim_{n \to \infty} \Phi_n d\mu_2 = \int_{\Omega_2} \lim_{n \to \infty} \left[\int_{\Omega_1} \varphi_n d\mu_1 \right] d\mu_2.$$

Since for every x_2, $\varphi_n(\cdot, x_2) \uparrow f(\cdot, x_2)$ and non-negative, measurable, so by applying MCT on $(\Omega_1, \mathcal{F}_1, \mu_1)$ we thus have

$$\lim_{n \to \infty} \left[\int_{\Omega_1} \varphi_n d\mu_1 \right] = \int_{\Omega_1} f d\mu_1.$$

Putting the previous equations together we obtain

$$\int_{\Omega_1 \times \Omega_2} f d\mu_1 \times \mu_2 = \int_{\Omega_2} \left[\int_{\Omega_1} f d\mu_1 \right] d\mu_2$$

and similarly

$$\int_{\Omega_1 \times \Omega_2} f d\mu_1 \times \mu_2 = \int_{\Omega_1} \left[\int_{\Omega_2} f d\mu_2 \right] d\mu_1$$

for any non-negative, measurable function f. The conclusions of the theorem follow immediately. ∎

8. Completion of product measure spaces. Recall that, if $(\Omega, \mathcal{F}, \mu)$ is a σ-finite measure space, then $\mathcal{F}^\mu$ is the completed σ-algebra of $\mathcal{F}$ under the measure μ, that is, $\mathcal{N}$ denotes the collection of all subsets of Ω with outer measure zero, then $\mathcal{F}^\mu = \sigma\{\mathcal{F}, \mathcal{N}\}$. We have shown that μ can be uniquely extended to a σ-finite measure on $\mathcal{F}^\mu$, denoted again by μ. Complications may arise if we consider the completion of $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \times \mu_2)$. In general, the completion of $\mathcal{F}_1 \otimes \mathcal{F}_2$ under $\mu_1 \times \mu_2$ does not coincide with the product σ-algebra of the completions of $\mathcal{F}_i$ under μ_i, but we have

Lemma 5.9 *Let $(\Omega_i, \mathcal{F}_i, \mu_i)$ be two σ-finite measure spaces. Then*

$$\mathcal{F}_1^{\mu_1} \otimes \mathcal{F}_2^{\mu_2} \subset (\mathcal{F}_1 \otimes \mathcal{F}_2)^{\mu_1 \times \mu_2}$$

and

$$(\mathcal{F}_1 \otimes \mathcal{F}_2)^{\mu_1 \times \mu_2} = (\mathcal{F}_1^{\mu_1} \otimes \mathcal{F}_2^{\mu_2})^{\mu_1 \times \mu_2}.$$

Proof. The proof is routine, left as an exercise. ∎

If $f : \Omega_1 \times \Omega_2 \to (-\infty, \infty)$ is measurable w.r.t $(\mathcal{F}_1 \otimes \mathcal{F}_2)^{\mu_1 \times \mu_2}$, then its section $f_{x_1} : \Omega_2 \to (-\infty, \infty)$ by sending x_2 to $f(x_1, x_2)$ is not necessary measurable w.r.t. $\mathcal{F}_2^{\mu_2}$, however, according to definition, there is a function $\tilde{f} : \Omega_1 \times \Omega_2 \to (-\infty, \infty)$ which is measurable w.r.t. $\mathcal{F}_1 \otimes \mathcal{F}_2$ and $f = \tilde{f}$ $\mu_1 \times \mu_2$-almost surely, and $\tilde{f}_{x_1}$ is measurable w.r.t. $\mathcal{F}_2$ for all $x_1 \in \Omega_1$. Moreover it is clear that $\tilde{f}_{x_1} = f_{x_1}$ for almost all $x_1 \in \Omega_1$ with respect to μ_1. Therefore f_{x_1} is $\mathcal{F}_2^{\mu_2}$-measurable for μ_1-almost all $x_1 \in \Omega_1$. The iterated integrals of f are defined to be those of $\tilde{f}$, and we can show that they are independent of the choice of a version $\tilde{f}$.

If $f \in L^1(\Omega_1 \times \Omega_2, (\mathcal{F}_1 \otimes \mathcal{F}_2)^{\mu_1 \times \mu_2}, \mu_1 \times \mu_2)$, then we choose $\tilde{f}$ which is $\mathcal{F}_1 \otimes \mathcal{F}_2$-measurable such that $f = \tilde{f}$ $\mu_1 \times \mu_2$-a.e., applying the Fubini theorem to $\tilde{f}$, we thus have the following refined version of Fubini's theorem.

Theorem 5.10 (Fubini's theorem) *Let $(\Omega_i, \mathcal{F}_i, \mu_i)$ be two σ-finite measure spaces. Suppose $f : \Omega_1 \times \Omega_2 \to (-\infty, \infty)$ is $(\mathcal{F}_1 \otimes \mathcal{F}_2)^{\mu_1 \times \mu_2}$-measurable.*

1) If $f \in L^1(\Omega_1 \times \Omega_2, (\mathcal{F}_1 \otimes \mathcal{F}_2)^{\mu_1 \times \mu_2}, \mu_1 \times \mu_2)$, then the two iterated integrals of f exist and coincide with the integral $\int_{\Omega_1 \times \Omega_2} f d(\mu_1 \times \mu_2)$.

2) Conversely, if one of the iterated integral of $|\tilde{f}|$ is finite, where $\tilde{f} = f$ $\mu_1 \times \mu_2$-a.e. and $\tilde{f}$ is $\mathcal{F}_1 \otimes \mathcal{F}_2$-measurable, then $f \in L^1(\Omega_1 \times \Omega_2, (\mathcal{F}_1 \otimes \mathcal{F}_2)^{\mu_1 \times \mu_2}, \mu_1 \times \mu_2)$.