FIN 508- MARTINGALE THROUGH MEASURE THEORY
LECTURE 3

LEBESGUE-STIELTJES MEASURES – OUTLINE OF CONSTRUCTION

These are the most important examples of measures used in analysis.

1. Increasing functions. Let $\rho : (a,b) \to (-\infty,\infty)$ be an increasing function, where $(a,b) \subset (-\infty,\infty)$ is an open interval. Then the left limit $\rho(t-) = \lim_{s\uparrow t} \rho(s)$ and the right limit $\rho(t+) = \lim_{u\downarrow t} \rho(u)$ exist at every $t \in (a,b)$, and

$$\rho(s) \leq \rho(t-) \leq \rho(t) \leq \rho(t+) \leq \rho(u)$$

for any $a < s < t < u < b$. ρ is called right continuous (resp. left continuous) at $t \in (a,b)$ if $\rho(t) = \rho(t+)$ (resp. $\rho(t) = \rho(t-)$). For any increasing function ρ on (a,b), $\rho_+(t) \equiv \rho(t+)$ is right continuous at every $t \in (a,b)$. ρ_+ is called the right continuous modification of ρ. Similarly, $\rho_-(t) = \rho(t-)$ is left continuous at any $t \in (a,b)$, ρ_- is called the left continuous modification of ρ. Therefore, an increasing function ρ is right continuous on (a,b) if ρ_+ coincides with ρ by definition.

2. Constructing Lebesgue-Stieltjes measure. For every right continuous increasing function ρ on (a,b) we construct a measure m_ρ on a σ-algebra $\mathcal{M}_\rho$ consisting of m_ρ-measurable subsets of (a,b). The construction is divided into several steps.

2.1) Decide what we want. Let $\mathscr{C}(a,b)$ be the π-system of all intervals $(s,t]$, where $a < s \leq t < b$, and we decide to assign a measure of such $(s,t]$ to be $m_\rho((s,t]) = \rho(t) - \rho(s)$.

2.2) Defining an outer measure. With m_ρ defined on the π-system $\mathscr{C}(a,b)$, we can assign an outer measure for any subset $E \subset (a,b)$, typically by

$$m_\rho^*(E) = \inf\left\{\sum_{j=1}^{\infty} m_\rho(C_j) : \text{ where } C_j \in \mathscr{C}(a,b) \text{ such that } \bigcup_{j=1}^{\infty} C_j \supset E\right\}$$

where the inf runs over all possible *countable* covers of E through $\mathscr{C}$. m_ρ^* is an outer measure on $\mathscr{P}(a,b)$ which is the σ-algebra of all subsets of (a,b).

2.3) Apply Caratheodory's theorem. By Theorem 2.4, the collection of all m_ρ^*-measurable subsets E of (a,b) is a σ-algebra on (a,b), denoted by $\mathcal{M}_\rho$, and $m_\rho^* : \mathcal{M}_\rho \to [0,\infty]$ is a measure. m_ρ is called the Lebesgue-Stieltjes measure on (a,b) associated with a right continuous increasing function ρ on (a,b).

The above three steps of constructing measures from outer measures apply to general cases, not only for measures on intervals. The most important question is of course to identify the measurable sets, i.e. to identify the σ-algebra $\mathcal{M}_\rho$ of m_ρ^*-measurable subsets.

2.4) Identifying measurable sets. Let $\mathscr{R}(a,b)$ be the ring of all subsets $E \subset (a,b)$ which are finite unions of subsets in $\mathscr{C}(a,b)$. The main technical step is to prove that m_ρ^* restricted on the ring $\mathscr{R}(a,b)$ is finitely additive. That is, if $E \in \mathscr{R}(a,b)$, so that $E = \cup_{j=1}^m C_j$ where $C_j = (s_j, t_j]$, $a < s_j \le t_j < b$ $(j = 1, 2, \cdots, m)$ such that $(s_j, t_j]$ are disjoint, then

$$m_\rho^*(E) = \sum_{j=1}^m \left(\rho(t_j) - \rho(s_j)\right).$$

Therefore, it follows that the outer measure m_ρ^* restricted on the ring $\mathscr{R}(a,b)$ is finitely additive.

We then can show that any set $E \in \mathscr{R}(a,b)$ is m_ρ^*-measurable, so that $\mathscr{C}(a,b) \subset \mathscr{R}(a,b) \subset \mathcal{M}_\rho$. Thus the Borel σ-algebra $\mathscr{B}(a,b) \subset \mathcal{M}_\rho$. It is easy to verify that

$$\begin{aligned}\mathscr{B}(a,b) &= (a,b) \bigcap \mathscr{B}(\mathbb{R}) = \left\{(a,b) \bigcap G : \text{ where } G \in \mathscr{B}(\mathbb{R})\right\} \\ &= \{G : \; G \subset (a,b) \text{ and } G \in \mathscr{B}(\mathbb{R})\}.\end{aligned}$$

Therefore any Borel subset of (a,b) is measurable with respect to the Lebesgue-Stieljes measure m_ρ. The restriction of the outer measure m_ρ^* on $\mathcal{M}_\rho$ is denoted by m_ρ.

Thus for every right-continuous increasing function ρ on an open interval (a,b), we have constructed a measure space $((a,b), \mathcal{M}_\rho, m_\rho)$, which is σ-finite and complete. Also $((a,b), \mathscr{B}(a,b), m_\rho)$ is a measure space, σ-finite, which is not complete in general.

3. Notations. If ρ is an increasing function on (a,b), then its right continuous modification $\rho_+(t) = \rho(t+)$ is right continuous, so that the Lebesgue-Stieljes measure m_{ρ_+} is defined, which is called the Lebesgue-Stieljes measure associated with ρ, denoted by m_ρ, that is, $m_\rho = m_{\rho_+}$ and $\mathcal{M}_\rho = \mathcal{M}_{\rho_+}$. In particular, m_ρ is the unique measure on $((a,b), \mathscr{B}(a,b))$ such that

$$m_\rho((s,t]) = \rho_+(t) - \rho_+(s) = \rho(t+) - \rho(s+)$$

for any $a < s < t < b$. In particular

$$\begin{aligned}m_\rho(\{t\}) &= \lim_{n\to\infty} m_\rho\left(\left(t - \frac{1}{n}, t\right]\right) = \lim_{n\to\infty}\left[\rho(t+) - \rho\left(\left(t - \frac{1}{n}\right)+\right)\right] \\ &= \rho(t+) - \rho(t-)\end{aligned}$$

for every $t \in (a,b)$. In particular, $\{t\}$ (where $t \in (a,b)$) is an m_ρ-null set if and only if ρ is continuous at t.