FIN 508- MARTINGALE THROUGH MEASURE THEORY
LECTURE 2

CARATH ´ EODORY ’ S EXTENSION THEOREM

In this section we review the main tools for constructing measures.

1. π-system and monotone class. Suppose $\mathcal{C}$ is a non-empty family of some subsets of Ω, then $\mathcal{C}$ is called a π-system if $\mathcal{C}$ is closed under the intersection, that is, $A \cap B \in \mathcal{C}$ whenever $A, B \in \mathcal{C}$. A collection$\mathcal{M}$ of some subsets of Ω is called a monotone class (or called a d-class) if 1) $\Omega \in \mathcal{M}$, 2) if $A, B \in \mathcal{M}$ and $A \subseteq B$ then $B \setminus A \in \mathcal{M}$, 3) $\cup_{n=1}^{\infty} A_n \in \mathcal{M}$ whenever $A_n \in \mathcal{M}$ such that $A_n \uparrow$.

Given a non-empty family $\mathcal{H}$ of some subsets of Ω, $\mathcal{M}(\mathcal{H})$ denotes the smallest monotone class which contains $\mathcal{H}$, called the monotone class generated by $\mathcal{H}$. The existence and uniqueness of $\mathcal{M}(\mathcal{H})$ are left as an exercise for the reader.

Lemma 2.1 (Dynkin's lemma) *If $\mathcal{C}$ is a π-system over Ω, then $\mathcal{M}(\mathcal{C})$ coincides with the smallest σ-algebra $\sigma(\mathcal{C})$ containing $\mathcal{C}$, that is, $\mathcal{M}(\mathcal{C}) = \sigma(\mathcal{C})$.*

Since a σ-algebra must be a monotone class, so that $\mathcal{M}(\mathcal{C}) \subseteq \sigma(\mathcal{C})$. To prove the other inclusion that $\sigma(\mathcal{C}) \subseteq \mathcal{M}(\mathcal{C})$, one only needs to verify that $\mathcal{M}(\mathcal{C})$ is a σ-algebra by using the fact that $\mathcal{C}$ is a π-system. The proof is routine, see for example page 193, D. Williams: Probability with martingales.

2. Uniqueness criterion. The following is a simple and useful uniqueness result.

Lemma 2.2 (Uniqueness lemma) *Suppose μ_j $(j = 1, 2)$ are two* finite *measures on a measurable space $(\Omega, \mathcal{F})$, and suppose $\mathcal{C} \subseteq \mathcal{F}$ is a π-system containing the sample space Ω such that $\sigma(\mathcal{C}) = \mathcal{F}$. If $\mu_1(E) = \mu_2(E)$ for every $E \in \mathcal{C}$, then $\mu_1 = \mu_2$ on $\mathcal{F}$.*

The proof of this lemma is an example how to use the Dynkin lemma.

Proof. Let $\mathcal{G}$ be the collections of all $E \in \mathcal{F}$ such that $\mu_1(E) = \mu_2(E)$. Then $\mathcal{C} \subseteq \mathcal{G}$ by assumptions. We prove that $\mathcal{G}$ is a monotone class. In fact, it is assumed that $\Omega \in \mathcal{G}$. Since $\mu_1(\varnothing) = \mu_2(\varnothing) = 0$ so that $\varnothing \in \mathcal{G}$. If $A, B \in \mathcal{G}$ and $A \subseteq B$, then, since $\mu_i(B) < \infty$, we have

$$\mu_1(B \setminus A) = \mu_1(B) - \mu_1(A) = \mu_2(B) - \mu_2(A) = \mu_2(B \setminus A)$$

which yields that $B \setminus A \in \mathcal{G}$. Suppose now $A_n \in \mathcal{G}$, and $A_n \uparrow$, then

$$\mu_1\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu_1(A_n) = \lim_{n\to\infty} \mu_2(A_n) = \mu_2\left(\bigcup_{n=1}^{\infty} A_n\right)$$

which implies that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{G}$. Thus $\mathcal{G}$ is a monotone class containing $\mathcal{C}$. By Lemma 2.1, $\mathcal{G} \supseteq \mathcal{M}(\mathcal{C}) = \sigma\{\mathcal{C}\} = \mathcal{F}$, so that $\mu_1 = \mu_2$ on $\mathcal{F}$. ∎

There is another version of the uniqueness for σ-finite measures.

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Lemma 2.3 *Let μ_j $(j = 1, 2)$ be two measures on $(\Omega, \mathcal{F})$, and $\mathcal{R} \subseteq \mathcal{F}$ be a ring such that $\sigma(\mathcal{R}) = \mathcal{F}$. Suppose μ_1 and μ_2 are σ-finite on $\mathcal{R}$: there is a sequence of subsets $G_n \uparrow \Omega$, $G_n \in \mathcal{R}$ and $\mu_1(G_n) = \mu_2(G_n) < \infty$ for every n. Suppose $\mu_1(E) = \mu_2(E)$ for every $E \in \mathcal{R}$. Then $\mu_1 = \mu_2$ on $\mathcal{F}$.*

Proof. Apply Lemma 2.2 to finite measures $\mu_j(\cdot \cap G_n)$ for every n to conclude that $\mu_1(E \cap G_n) = \mu_2(E \cap G_n)$ for every n and $E \in \mathcal{F}$. Letting $n \uparrow \infty$ to obtain that $\mu_1(E) = \mu_2(E)$ for every $E \in \mathcal{F}$. ■

3. Measurable sets and Caratheodory's extension theorem. The construction of measures rely on the extension theorem of Carathéodory's, a theorem that tells us how to select measurable subsets for an outer measure. Let $\mathcal{H}$ be a σ-algebra over a sample space Ω, and $\mu^* : \mathcal{H} \to [0, \infty]$ be an outer measure on $(\Omega, \mathcal{H})$, so that

3.1) $\mu^*(\varnothing) = 0$;

3.2) $\mu^*(A) \leq \mu^*(B)$ for any $A \subseteq B$, $A, B \in \mathcal{H}$; and

3.3) μ^* is countably sub-additive:

$$\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$$

for any sequence $E_n \in \mathcal{H}$ $(n = 1, 2, \cdots)$.

A subset $E \in \mathcal{H}$ is called μ^*-measurable, if E satisfies the Carathéodory condition that

$$\mu^*(F) = \mu^*(F \cap E) + \mu^*(F \cap E^c) \quad \text{for every } F \in \mathcal{H}. \tag{2.1}$$

The collection of all μ^*-measurable subsets is denoted by $\mathcal{M}$ or $\mathcal{M}(\mathcal{H}, \mu^*)$ (in order to indicate the dependence on the outer measure μ^* on $(\Omega, \mathcal{H})$.)

Theorem 2.4 *(*Caratheodory*) Let $(\Omega, \mathcal{H})$ be a measurable space and μ^* be an outer measure on $(\Omega, \mathcal{H})$. Then the collection $\mathcal{M}(\mathcal{H}, \mu^*)$ of all μ^*-measurable subsets forms a σ-algebra over Ω, and μ^* restricted on $\mathcal{M}(\mathcal{H}, \mu^*)$ is a measure.*

The proof of the previous theorem is exactly the same as that in Part A Integration.

Theorem 2.5 *(*Caratheodory*'s extension theorem) Let Ω be a space and $\mathcal{R}$ be a σ-algebra. If μ is a measure on the algebra $\mathcal{R}$, the outer measure μ^* is defined by*

$$\mu^*(E) = \inf\left\{\sum_{j=1}^{\infty} \mu(E_j) : \text{ where } E_j \in \mathcal{R} \text{ and } \bigcup_{j=1}^{\infty} E_j \supseteq E\right\}$$

where the inf runs over all countable cover $\{E_j\}$ of E and $E_j \in \mathcal{R}$. Then any set $E \in \mathcal{R}$ is μ^*-measurable, and $\mu^*(E) = \mu(E)$, so that μ^* restricted on the σ-algebra of all μ^*-measurable subsets is an extension of μ.

This is a consequence of Theorem 2.4, the only thing need to check is that every element E of $\mathcal{R}$, $\mu^*(E) = \mu(E)$ (which is direct but not trivial).

4. Null sets. A subset $E \in \mathcal{H}$ is μ^*-null set if $\mu^*(E) = 0$. If $\{E_i : i = 1, 2, \cdots\}$ is a sequence of μ^*-null sets, so is $\bigcup_{i=1}^{\infty} E_i$ by the countable sub-additivity. By definition, any μ^*-null set is μ^*-measurable. Therefore μ^* is a *complete* measure on $(\Omega, \mathcal{M}(\mathcal{H}, \mu^*))$.

5. Completion of a measure space. If $(\Omega, \mathcal{F}, \mu)$ is a measure space, so it is extended to an outer measure μ^* defined by

$$\mu^*(E) = \inf \left\{ \sum_{j=1}^{\infty} \mu(E_j) : \text{ where } E_j \in \mathcal{F} \text{ such that } \bigcup_{n=1}^{\infty} E_j \supset E \right\}$$

and let $\mathcal{F}^*$ be the σ-field of all μ^*-measurable subsets. Then $(\Omega, \mathcal{F}^*, \mu)$ is a measure space, and $\mathcal{F} \subseteq \mathcal{F}^*$. Let $\mathcal{N}^\mu$ denotes the collection of all μ^*-null subsets, so that $\mathcal{N}^\mu \subseteq \mathcal{F}^*$ too. Hence $\mathcal{F}^\mu \equiv \sigma \{\mathcal{N}^\mu, \mathcal{F}\} \subseteq \mathcal{F}^*$. Thus $(\Omega, \mathcal{F}^\mu, \mu)$ is a complete measure space, called the completion of $(\Omega, \mathcal{F}, \mu)$.