FOURIER SERIES AND PDEs
LECTURE 5

THE HEAT EQUATION

Derivation in one space dimension

A straight rigid metal rod lies along the x-axis. The lateral surface is insulated to prevent heat loss.

Let ρ be the mass density per unit length, c be the specific heat, $T(x,t)$ be the temperature and:

- $+q(x,t)$ be the heat flux from $-$ to $+$;
- $-q(x,t)$ be the heat flux from $+$ to $-$.

Consider any interval $[a, a+h]$:

$$\text{internal energy} = \int_a^{a+h} \rho c T(x,t)\,\mathrm{d}x; \tag{3.1}$$

$$\text{net heat flux out of } [a, a+h] = q(a+h,t) - q(a,t). \tag{3.2}$$

By conservation of energy, for every interval $[a, a+h]$,

$$\text{rate of change of internal energy} + \text{net heat flux out} = 0. \tag{3.3}$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^{a+h} \rho c T(x,t)\,\mathrm{d}x + [q(a+h,t) - q(a,t)] = 0. \tag{3.4}$$

Hence, by Leibniz,

$$\frac{1}{h}\int_a^{a+h} \rho c \frac{\partial T}{\partial t}(x,t)\,\mathrm{d}x + \left[\frac{q(a+h,t) - q(a,t)}{h}\right] = 0, \tag{3.5}$$

and on letting $h \to 0$ we get the equation

$$\rho c \frac{\partial T}{\partial t} + \frac{\partial q}{\partial x} = 0. \tag{3.6}$$

In order to close the system, we need to describe how the heat flux varies as a function of x, t and T.

In one space dimension, the law of heat conduction, also known as Fourier's law, states that the time rate of heat transfer through a material is proportional to the negative gradient in the temperature:

$$q(x,t) = -k\frac{\partial T}{\partial x}, \tag{3.7}$$

where k is the *thermal conductivity*. The negative sign reflects the fact that heat flows from high temperatures to low temperatures.

On substituting from equation (3.7) into (3.6) we arrive at the heat equation in one space dimension:

$$\frac{\partial T}{\partial t} = \kappa\frac{\partial^2 T}{\partial x^2}, \tag{3.8}$$

where $\kappa = k/\rho c$ is the *thermal diffusivity*.

Units and nondimensionalisation

Consider the units of the variables (x, t and T) and parameter (κ) associated with the heat equation. We will use the following notation to denote the dimensions of a variable or parameter:

$$[p] = \text{dimensions of } p. \tag{3.9}$$

In SI units we have

$$[x] = \text{m (metres)}, \quad [t] = \text{s (seconds)}, \quad [T] = \text{K (Kelvin)}. \tag{3.10}$$

For the heat equation,

$$\frac{\partial T}{\partial t} = \kappa\frac{\partial^2 T}{\partial x^2}, \tag{3.11}$$

we see that the left-hand side has units Ks^{-1}. The term $\partial^2 T/\partial x^2$ has units Km^{-2}. Hence for the units of the right-hand side to balance those of the left-hand side, the units of κ must be $[\kappa] = \text{m}^2\text{s}^{-1}$.

We can *non-dimensionalise* the heat equation by scaling our variables and parameters. For example, let

$$x = l\hat{x}, \quad t = \tau\hat{t}, \quad T = T_0\hat{T}, \tag{3.12}$$

where l, τ and T_0 are a typical lengthscale, timescale and temperature, respectively, for the problem under consideration. Then

$$\frac{\partial}{\partial t} = \frac{\mathrm{d}\hat{t}}{\mathrm{d}t}\frac{\partial}{\partial \hat{t}} = \frac{1}{\tau}\frac{\partial}{\partial \hat{t}}, \tag{3.13}$$

$$\frac{\partial}{\partial x} = \frac{\mathrm{d}\hat{x}}{\mathrm{d}x}\frac{\partial}{\partial \hat{x}} = \frac{1}{l}\frac{\partial}{\partial \hat{x}}, \tag{3.14}$$

$$\frac{\partial^2}{\partial x^2} = \frac{\mathrm{d}\hat{x}}{\mathrm{d}x}\frac{\partial}{\partial \hat{x}}\left(\frac{1}{l}\frac{\partial}{\partial \hat{x}}\right) = \frac{1}{l^2}\frac{\partial^2}{\partial \hat{x}^2}, \tag{3.15}$$

and substituting into the heat equation we have

$$\frac{T_0}{\tau}\frac{\partial \hat{T}}{\partial \hat{t}} = \frac{\kappa T_0}{l^2}\frac{\partial^2 \hat{T}}{\partial \hat{x}^2}. \tag{3.16}$$

Rearranging gives

$$\frac{\partial \hat{T}}{\partial \hat{t}} = \frac{\kappa\tau}{l^2}\frac{\partial^2 \hat{T}}{\partial \hat{x}^2}. \tag{3.17}$$

Considering the problem on a timescale where $\tau = l^2/\kappa$ gives

$$\frac{\partial \hat{T}}{\partial \hat{t}} = \frac{\partial^2 \hat{T}}{\partial \hat{x}^2}. \tag{3.18}$$

Notice that now

$$[\hat{x}] = 1, \quad [\hat{t}] = 1, \quad [\hat{T}] = 1, \tag{3.19}$$

since

$$[l] = \mathrm{m}, \quad [\tau] = \left[\frac{l^2}{\kappa}\right] = \mathrm{s}, \quad [T_0] = \mathrm{K}. \tag{3.20}$$

This means that we can compare heat problems on different scales: for example, two systems with different l and κ will exhibit comparable behaviour on the same time scales if l^2/κ is the same in each problem.

Heat conduction in a finite rod

Let the rod occupy the interval $[0, L]$. If we look for solutions of the heat equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}, \tag{3.21}$$

which are separable, $T(x,t) = F(x)G(t)$, we find that

$$\underbrace{\frac{\kappa}{F(x)} F''(x)}_{\text{independent of } t} = \underbrace{\frac{1}{G(t)} G'(t)}_{\text{independent of } x}, \tag{3.22}$$

and hence both sides are constant (independent of both x and t). If the constant is $-\kappa\lambda^2$, $F(x)$ satisfies the ODE

$$F''(x) = -\lambda^2 F(x), \tag{3.23}$$

the solution of which is

$$F(x) = A\sin(\lambda x) + B\cos(\lambda x). \tag{3.24}$$

If the ends are held at zero temperature then $F(0) = F(L) = 0$. The boundary condition at $x = 0$ gives $B = 0$ and the boundary condition at $x = L$ gives

$$A\sin(\lambda L) = 0. \tag{3.25}$$

Since we want $A \neq 0$ to avoid a non-trivial solution, it must be that $\sin(\lambda L) = 0$ *i.e.* λ must be such that $\lambda L = n\pi$ where n is a positive integer. Hence λ must be one of the numbers

$$\left\{ \frac{n\pi}{L} : n = 1, 2, 3, \ldots \right\}. \tag{3.26}$$

Moreover $G(t)$ satisfies the ODE

$$G'(t) = -\kappa\lambda^2 G(t) = -\frac{\kappa n^2 \pi^2}{L^2} G(t), \tag{3.27}$$

and, therefore, $G(t) \propto e^{-n^2\pi^2\kappa t/L^2}$. Hence we have the separable solution

$$T_n(x,t) = a_n \sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2\kappa t/L^2}, \tag{3.28}$$

which satisfies the heat equation (3.8) and the boundary conditions

$$T(0,t) = 0 \text{ and } T(L,t) = 0 \text{ for } t > 0. \tag{3.29}$$

The general solution can therefore be written as a linear combination of the T_n so that

$$T(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2\kappa t/L^2}. \tag{3.30}$$