Third week lessons.

(Divided into 3 lectures of 50 minutes each)

Lecture – 7 (50 minutes)

a) Absolute value
b) Establishment of $|x| < a$ implies $-a < x < a$ and its converse.
c) Simplification removing the absolute sign and using absolute sign.

A. Absolute value.

Let x denote any real number. The absolute value (or modulus or numerical value) of x, denoted by |x|, is defined by

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Clearly, $|x| \geq 0$. [if the value of x is negative, we take –x which gives the positive result.]

By the definition , we can write $|x|^2 = x^2$ for all positive and negative real values of x. Also $-x \leq |x|$ and $x \leq |x|$ for all values of x.

Theorem 1:

For real numbers x and y,

i) $|x + y| \leq |x| + |y|$
ii) $|x - y| \geq |x| - |y|$

Proof:

(i) $|x + y|^2 = (x + y)^2$

$\Rightarrow |x + y|^2 = x^2 + 2xy + y^2$

$\Rightarrow |x + y|^2 = |x|^2 + 2xy + |y|^2$

$\leq |x|^2 + 2|x||y| + |y|^2$

[For $x \leq |x|$ and $y \leq |y|$]

$\Rightarrow |x + y|^2 \leq (|x| + |y|)^2$

Hence $|x + y| \leq |x| + |y|$ Proved.

(ii) $|x| = |x - y + y| \leq |x-y| + |y|$

[using theorem (i)]

$\Rightarrow (|x| - |y|) \leq |x - y|$ proved.

B. Establishment of |x|< a implies –a < x < a and its converse.

Theorem 2:

(i) If a is a positive real number, then $|x|<a$ implies $-a < x < a$.

(ii) If $-a<x<a$ then $|x|<a$.

Proofs:

(i) We know that

$-x \leq |x|$ and $x \leq |x|$

$\Rightarrow -x \leq |x| < a$ and $x \leq |x| < a$

[For $|x| < a$, given]

$\Rightarrow -x < a$ and $x < a$

$\Rightarrow -x < a$ and $x < a$

Hence $-a<x<a$. Proved.

(ii) $-a<x<a$

$\Rightarrow -a < x$ and $x < a$.

$\Rightarrow -x < a$ and $x < a$.

Hence $|x|< a$. Proved.

C. Simplification removing the absolute sign and using absolute sign.

We can apply the above proved theorem 2 to use and remove the absolute sign in inequalities which will be clear by the examples given below.

Example 1:

Rewrite without absolute value sign:

a) $|x|<3$ b) $|x - 5| < 2$ c) $|3x + 2| < 1$

Here,

a) $|x| < 3$

$\Rightarrow -3 < x < 3$.

b) $|x-5| <2$

$\Rightarrow -2 < (x - 5) < 2$

$\Rightarrow -2 + 5 < x < 2 + 5$.

[adding 5 to each side]

$\Rightarrow 3 < x < 7$

c) $|3x + 2| < 1$

$\Rightarrow -1 < (3x + 2) < 1 \Rightarrow -1 - 2 < 3x < 1 - 2$

$\Rightarrow -3 < 3x < -1$

$\Rightarrow -1 < x < -1/3$

Example 2:

Rewrite the following by using the absolute sign:

a) $-5 < x < 5$ b) $-3 < x < 5$

c) $-4 < x < -1$

Here,

a) $-5 < x < 5$

$\Rightarrow |x| < 5$

b) $-3 < x < 5$

$\Rightarrow -3 - 1 < x - 1 < 5 - 1$

$\Rightarrow -4 < x - 1 < 4 \Rightarrow |x - 1| < 4.$

c) $-4 < x < -1$

$\Rightarrow -8 < 2x < -2$

[multiplying each side by 2]

$\Rightarrow -8 + 5 < 2x + 5 < -2 + 5$

$\Rightarrow -3 < 2x + 5 < 3$

$\Rightarrow |2x + 5| < 3$

Note: For the problems of type 2, we need to make a positive number in right of x and the same number with negative sign in left of x. After getting such result, we use theorem 2(ii). If the numbers in the ends are of different magnitudes, then we have to add or subtract a number so that they will be of same magnitudes. To find that number which must be added or subtracted, first find the sum of the right and left numbers then divide the sum by 2. The resulting number is the required number which must be added or subtracted.

In example 2(b), the sum of end numbers is 2, dividing this sum by 2 we get 1, so 1 must be subtracted from each term of the inequality.

Note: In example 2(c), the sum of end numbers is -5 which is an odd number, not divisible by 2. So, we have to multiply each term by 2 to get the sum of end numbers an even number. After this the sum of end numbers is -10, dividing the sum by 2 we get -5, so 5 must be added to each term.

Lecture – 8 (50 minutes)

a) Exercise of the problems related to the real number system.
b) Assignment

A. Exercise of the problems related to the real number system.

1. If $-7 \leq (2x + 5) \leq 7$, prove that $-6 \leq x \leq 1$.
2. If (i) $x = 2, y = 3$ (ii) $x = -5, y = 4$

   Verify that:

   a) $|x + y| \leq |x| + |y|$

   b) $|x - y| \geq |x| - |y|$
3. Find the value of:

   a) $|-2| + 4$ b) $3 - |-5|$ c) $|-5| + |-2|$
4. Rewrite the following without absolute value sign:

   a) $|x - 3| < 2$ b) $|3x + 5| \leq 1$

   c) $|2x + 5| \leq 3$

5. Rewrite the following using absolute value sign:

   a) $-5 < x < 7$ b) $-3 \leq x \leq -1$

   c) $-6 < x < 1$ d) $-5 \leq x \leq 2$

   e) $-4 < x < 2$ f) $-1 < x < 4$
9. Prove that

   a) $|x - y| + |y - z| > |x - z|$

   b) $|x| + |y| \geq |x - y|$

B. Assignment:

1. Prove that $\sqrt{7}$ is not a rational number.

2. Prove that $\sqrt{3}$ is an irrational number.

3. Show that $\frac{2a + 3b}{a + b}$ is a rational number lying between 2 and 3, where a and b are positive rational numbers. [Hints: $\frac{2a + 3b}{a + b} = 2 + \frac{b}{a + b}$ and $\frac{2a + 3b}{a + b} = 3 - \frac{a}{a + b}$]

4. Find the value of:

   i) $|3| + |-2| - |-3|$

   ii) $|-6| + |-5| - |8|$

3. Solve: i) $|x – 5| = 3$ ii) $|5 – 2x| = 9$ iii) $|10 –2x| = |x + 5|$
   iv) $|x + 7| = |x – 5|$

[Hints: If $|x| = a$ then $x = a$ or $–a$, and $|x| = |y| \Rightarrow x = \pm y$ Get the final values of x i) 8, 2 ii) –2, 7 iii) 5/3, 15 iv) –1]

4. Solve: i) $|4 – 2x| < 2$ ii) $|2x – 3| > 5$

   iii) $|y + 1| \le -9$ iv) $|2x + 3| < 5$

   [iii) no solution(why?), Because absolute value cannot be less than a negative number] iv)-4<x-1]

5. Write using absolute sign: i) $–2 < x < 4$ ii) $–1 < x < 4$ iii) $–2 < x < -1$.

6. If A = [-2, 2) and B = (-1, 3] find the value of A – B, $A \cap B$ and $A \cup B$.

7. Solve: i) $x < x + 5$ ii) $2x –5 \ge 3x + 2$ [Ans: i) {x: x is a real number} ii) $x \le 7$]

8. Define absolute value. Prove that $|x|<a$ implies $–a < x < a$. Also prove that $–a < x < a$ implies $|x| < a$.

9. Prove that $|x + y| \le |x| + |y|$ and $|x – y| \ge |x| - |y|$.

10. Rewrite without using absolute sign: $|2x + 3| \le 8$.

11. If (i) x = 2, y = 3 (ii) x = -5, y = 4 Verify the following

   i) $|x + y| \le |x| + |y|$ ii) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ iii) $|x - y| \ge |x| - |y|$

12. If $-10 < 5x + 10 < 5$, prove that $-4 < x < -1$

Lecture – 9 (50 minutes)

Permutation and combination

a) Basic principles of counting
b) Factorial notation
c) Class works

A. Basic Principles of counting

If an event can occur in m different ways and if following it, a second event can occur in n different ways, then the two events in succession can occur in $m \times n$ different ways.

Example1:

Three persons enter a railway carriage, where there are 7 vacant seats. in how many ways can they seat themselves ?

Solution :

There are 7 ways in which the first person can seat himself. Now, the second person can occupy any of the remaining 6 seats. So, he can be seated in 6 ways. Similarly, the third can occupy a sit in 5 ways.
Hence by the principle of counting,
the total number of ways = $7 \times 6 \times 5 = 210$

Example 2:

How many numbers are there between 100 and 1000 such that every digit is either 2 or 9
Solution :
Clearly, the number will be of 3 digits. The repetition of the digits 2 and 9 is allowed.
$\therefore$ Units place can be filled by 2 ways.

Tens place can be filled by 2 ways.

Hundreds place can be filled by 2 ways.

Hence, the total numbers formed = $2 \times 2 \times 2 = 8$

Example 3:
How many numbers are there between 100 and 1000 such that 7 is in the unit's place?

Solution :

We have to form 3-digit numbers with 7 at the units' place. Zero cannot be at the hundred's place.
$\therefore$ Hundred's place can be filled in 9 ways.

Tens place can be filled in 10 ways.
Units place can be filled in 1 way.
$\therefore$ The total numbers formed = $9 \times 10 \times 1 = 90$

B. Factorial notation

The factorial is represented by ! after a non negative integer. For example 5! is read as "Five factorial" which gives the product $5\times4\times3\times2\times1 = 120$. The generalized formula for n! is $n! = n\times(n-1)\times(n-2)\times....\times3\times2\times1$.

It should be noted that $0! = 1$. and $8! = 8\times7!$ or $8\times7\times6!$ or $8\times7\times6\times5!$ and so on.

C. Class work

Use the principle of counting to solve the following problems:

1. There are 10 buses running between Janakpur and Kathmandu. In how many ways can a man go from Janakpur to Kathmandu and return by a different bus ?
2. In a class there are 27 boys and 14 girls. The teacher wants to select 1 boy and I girl to

represent the class in a function. In how many ways can the teacher make this selection ?

3. There are 6 multiple choice questions in an examination. How many sequences of answers are possible if first three questions have 4 choices each and the next three have 5 each ?

4. Write the values of i) 5! ii) 9! iii) $\frac{8!}{5!}$ iv) $\frac{102!}{99!}$ v) $\frac{458!}{456!}$