

Fourth week lessons.

(Divided into 3 lectures of 50 minutes each)

Lecture – 10 (50 minutes)

- a) Permutation of n different objects, taken all or some(r) of them.
- b) Permutation of n objects containing some repeated objects.
- c) Class works

A. Permutation of n different objects, taken all or some of them.

The different arrangements which can be made out of a given number of things by taking some or all at a times, are called permutations.

Let r and n be the positive integers such that $1 \leq r \leq n$. Then the numbers of all permutations of n things taken r at a time is denoted by $P(n, r)$ or ${}^n P_r$.

Theorem

Let $1 \leq r \leq n$. Then the number of all permutations of n different things taken r at a time is given by

$$P(n, r) = \frac{n!}{(n-r)!}$$

Proof:

The number of all permutations of n things taken r at a time is the same as the number of different ways in which r places in a row can be filled with n different things.

The first place can be filled up by any one of these n things. So, there are n ways of filling up the first place.

We are left with $(n - 1)$ things. So, there are $(n - 1)$ ways of filling up the second place.

Now, we are left with $n - 2$ things. So there are $n - 2$ ways of filling up the third place.

By the fundamental principle of counting, the number of ways of filling up the first three places is

$$n (n - 1) (n - 2).$$

Continuing in this manner, the r^{th} place can be filled up with any of these $n - (r - 1)$ things. So there are $n - r + 1$ ways of filling up the r^{th} place.

Thus, by the fundamental principle of counting the total number of ways is

$$\begin{aligned}
 P(n, r) &= n(n-1)(n-2) \dots (n-r+1) \\
 &= \frac{n(n-1)(n-2) \dots (n-r+1) \cdot (n-r) \cdot \dots \cdot 3 \cdot 2 \cdot 1}{(n-r)(n-r-1) \dots 3 \cdot 2 \cdot 1} \\
 &= \frac{n!}{(n-r)!}
 \end{aligned}$$

Cor 1

The number of all permutations of n different things, taken all at a time is given by $P(n, n) = n!$

Cor 2

Prove that $0! = 1$

Proof:

$$p(n, r) = \frac{n!}{(n-r)!}$$

or
$$p(n, n) = \frac{n!}{(n-n)!}$$

or
$$n! = \frac{n!}{0!}$$

or
$$0! = \frac{n!}{n!}$$

$$= 1 \qquad \text{proved}$$

Example 1:

How many words can be formed from the letters of the word 'SECONDARY' so that
 (i) the vowels always come together;

(ii) the vowels are never together.

Solution :

i) When the vowels E, O, A are always together, their group can be treated as one object. Then the objects to be arranged are S, C, N, D, R, Y, (AOE). These 7 objects can be arranged in $P(7,7) = 7! = 5040$ ways.

But, corresponding to each of these arrangements, the vowels A, O, E can be arranged amongst themselves in $P(3,3) = 3! = 6$ ways.

\therefore Required number of ways = $5040 \times 6 = 30240$

ii) Total number of words = $P(9, 9) = 362880$

The number of words in which vowels are never together
 = Total arrangements - vowels are together
 = $362880 - 30240 = 332640$

Example 2

Find the number of ways in which 5 boys and 3 girls can be seated in a row so that no two girls are together.

Solution :

The 5 boys may occupy 5 places in $P(5, 5) = 5! = 120$ ways.

Since no two girls are to sit together, we may arrange the three girls at the 6 places as shown (by dots) below.

....B B.... BBB

Total number of arrangements for 3 girls
 = $P(6, 3)$
 = $6 \times 5 \times 4 = 120$

∴ Total arrangements of boys and girls
 = $120 \times 120 = 14400$

[Note: this problem can also be solved by the method used in example1, second part]

B. Permutation of n objects containing some repeated objects.

Let there be objects, of which p objects are alike of one kind, q are alike of a second kind, r are alike of a third kind, and the rest different, then the number of permutations of the n things taken all together equals

$$\frac{n!}{p!q!r!}$$

Example

- i) Find how many arrangements can be made with the letters of the word MATHEMATICS?
- ii) In how many of them, the vowels are together?

Solution :

- (i) There are 11 letters in the word 'MATHEMATICS', out of these letters M occurs twice, A occurs twice, T occurs twice and the rest are all different.

Required number of arrangements is

$$= \frac{11!}{2!.2!.2!}$$

$$= 4989600$$

- i) The given word contains 4 vowels A, E, A, I. Treating these 4 vowels as one object, we have to arrange 8 letters; out of which M occurs twice, T occurs twice and the rest are all different.

$$\text{The number of arrangements} = \frac{8!}{2!.2!}$$

$$= 10080$$

Again, the four vowels in which A occurs twice and the rest are all different, can be arranged amongst themselves in $\frac{4!}{2!} = 12$ ways.

∴ Total number of arrangements in which vowels are together

$$= 10080 \times 12$$

$$= 120960$$

C. Class work

1. In how many ways can 10 people line up at a ticket window of a cinema hall ?
2. Find the number of words formed (may be meaningless) by using all the letters of word 'EQUATION', using each letter exactly once.
3. Seven students are contesting election for the president of the students union. In how many ways can their names be listed on the ballot paper?
4. Find the number of permutations of the letters of the word 'ENGLISH'. How many of these begin with E and end with I ?
5. In how many ways can the letters of the word 'FAILURE' be arranged so that the consonants may occupy only odd position ?
6. How many permutations can be formed by the letters of the word 'VOWELS' when
 - i) the letters can be placed anywhere.
 - ii) each word begins with E.
 - iii) each word begins with O and ends with L.
 - iv) all vowels come together.
 - v) all consonants come together.
7. There are 3 blue balls, 4 red balls and 5 green balls. In how many ways can they be arranged in a row ?
8. How many 7-digit numbers can be formed using the digits 1, 2, 0, 2, 4, 2 and 4 ?

9. How many numbers can be formed with the digits 1, 2, 3, 4, 3, 2, 1 so that the odd digits always occupy the odd places ?

Lecture – 11 (50 minutes)

- a) Permutation of objects when an object can be placed repeatedly.
- b) Circular permutation.

A. Permutation of objects when an object can be placed repeatedly.

The number of permutations of n different objects, taken r at a time when each may be repeated any number of times in each arrangement is n^r .

Example:

How many 4-digit numbers can be formed with the digits 1, 2, 3, 4, 5, 6 when a digit may be repeated any number of times in any arrangement?

Solution :

Since there is no restriction on repetition of digits, each one of the thousand's, hundred's, ten's and unit's digits can be filled in 6 ways.

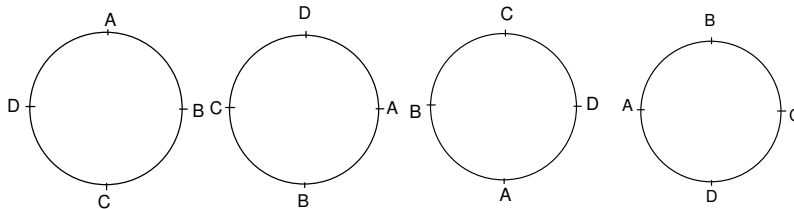
$$\therefore \text{Required number of numbers} = 6.6.6.6 = 6^4 = 1296.$$

B. Circular Permutations

The arrangements of objects in a line are known as linear permutations. If we arrange them in the form of a circle, we call them, circular permutations.

If we consider the linear permutations,

ABCD, BCDA, CDAB and DABC then, clearly they are distinct. Now, we arrange A, B, C, D along the circumference of a circle as shown below:



If we consider the position of an object relative to others, then we find that the above four arrangements are the same. So the number of arrangements in a circle is less than that in a line. By the diagram given above, the four arrangements ABCD, BCDA, CDAB and DABC in a circle are the same while they are different in a linear arrangement.

Note that there are two types of circular permutations.

- 1 The circular permutations in which clockwise and the anticlockwise arrangements give rise to different permutations, e.g. seating arrangements of persons round a table.
2. The circular permutations in which clockwise and the anticlockwise arrangements give rise to same permutations. e.g. Arranging some beads to form a necklace.

Theorem

The number of circular permutations of n different objects is $(n - 1)!$

Proof:

Each circular permutation corresponds to n linear permutations depending upon from where we start. Since there are n! linear permutations, it follows that there are

$$\frac{n!}{n} = (n - 1) ! \text{ Circular permutations.}$$

Cor 1

The number of ways in which n different beads can be arranged to form a necklace is

$$\frac{(n - 1)!}{2}$$

Example

In how many ways can a party of 4 men and 4 women be seated at a circular table so that no two women are adjacent?

Solution :

The four men be seated at the circular table such that there is a vacant seat between every pair of men. The number of ways in which these 4 men can be seated at the circular table = $(4 - 1) ! = 3 ! = 6$

Now, the four vacant seats may be occupied by four women in $P (4, 4) = 4 ! = 24$ ways

∴ Required number of ways = $6 \times 24 = 144$

Lecture – 12 (50 minutes)

- a) Difference between permutation and combination.
- b) Combination of r objects selected from n objects.
- c) Restricted (included or excluded) combinations.

A. Difference between permutation and combination.

The different groups or collection that can be formed out of a given set of things by taking some or all of them at a time (without regard to the order of their arrangements) are called their combinations.

For example: the combination of letters A, B, C taken two at a time are AB, BC, CA. In case of combination order does not make any sense. So, AB is same as BA. In permutation, AB and BA are different.

We can distinguish permutation and combination by the following points

1. In combination, only the selection of objects is made whereas in permutation, not only the selection is made but also an arrangement in definite order is considered.
2. The ordering of the selected objects is of no significance in combination whereas in permutation the ordering is essential.
3. Usually, the number of permutation is more than the number of combinations.

B. Combination of r objects selected from n objects.

Theorem

The number of all combinations of n distinct objects, taken r at a time, is given by

$$C(n, r) = \frac{n!}{(n-r)!r!}$$

Proof:

Let $C(n, r) = x$

Each combination contains r objects which may be arranged amongst themselves in r! ways.

Thus, each combination gives rise to r! permutations.

That is, $P(n, r) = x \times r!$

Or $P(n, r) = C(n, r) \times r!$

$$\therefore C(n, r) = \frac{n!}{(n-r)!r!} \quad \text{proved.}$$

Example 1:

An examination paper containing 12 questions consists of two parts A and B. Part A contains 7 questions and part B contains 5 questions. A candidate is required to attempt 8 questions, selecting at least 3 from each part. In how many ways can the candidate select the questions?

Solution

The candidate may select the questions as under.

Part A(7)		Part B(5)
C(7, 3)	and	C(5, 5)
or C(7, 4)	and	C(5, 4)
or C(7, 5)	and	C(5, 3)

The number of ways of selecting the questions.

$$= C(7, 3) \times C(5, 5) + C(7, 4) \times C(5, 4) + C(7, 5) \times C(5, 3)$$

$$= 35 + 175 + 210$$

$$= 420$$

[Note: generally and is replaced by \times and or is replaced by $+$ as in the above example]

Example 2:

In an examination, a candidate has to pass in each of the 5 subjects. In how many ways can he fail?

Solution :

The candidate can fail in 1 or 2 or 3 or 4 or 5 subjects out of 5 in each case.

Total number of ways in which he can fail

$$\begin{aligned} &= C(5, 1) + C(5, 2) + C(5, 3) + C(5, 4) + C(5, 5) = 5 + 10 + 10 + 5 + 1 \\ &= 31 \end{aligned}$$

C. Restricted (included or excluded) combinations.

The number of combinations of n different objects taken r at a time in which

a) m particular objects are excluded $= {}^{n-m}C_r$

b) m particular objects are included $= {}^{n-m}C_{r-m}$

Example:

Find the number of ways in which a student can choose 5 courses out of 9 courses, when 2 courses are compulsory?

Solution:

Here $n = 9$, $r = 5$ and $m = 2$

$$\text{Required number of selections} = {}^{9-2}C_{5-2} = {}^7C_3 = 35.$$