Seventh week lessons

Function (continued)

(Divided into 3 lectures of 50 minutes each)

Lecture – 19 (50 minutes)

a) Domain and range of the functions of the type f(x) = $\sqrt{x^2 - a^2}$ and f(x) = $\sqrt{a^2 - x^2}$
b) Class works

A. Domain and range of the functions of the type f(x) = $\sqrt{x^2 - a^2}$ and f(x) = $\sqrt{a^2 - x^2}$.

As mentioned in the above note (Lecture- 17), the expression under the square root must be positive. Using that rule, we find domain. The range of the square root function cannot contain negative numbers. So we take positive numbers only.

Further note that $x^2 \le a^2$ implies $|x| \le a$ and hence $-a \le x \le a$.

And $x^2 \ge a^2$ implies $|x| \ge a$ which gives $-a \ge x$ and $x \ge a$ separately. Let us see some examples.

Example1: Find the domain and range of y = $\sqrt{9 - x^2}$

Solution:

For domain,

the expression $\sqrt{9 - x^2}$ is well defined if

$9 - x^2 \ge 0$

or $x^2 - 9 \le 0$

or $x^2 \le 9$

or $|x| \le 3$

or $-3 \le x \le 3$

Hence domain is [-3, 3]

For range,

$y = \sqrt{9 - x^2}$

or $y^2 = 9 - x^2$

or $x = \sqrt{9 - y^2}$

here $\sqrt{9 - y^2}$ is well defined if

$9 - y^2 \ge 0$

or $y^2 - 9 \le 0$

or $y^2 \le 9$

or $|y| \le 3$

or $-3 \le y \le 3$

But the given function contains the positive square root $\sqrt{9-x^2}$. So the range contains only positive numbers.
Hence the range is $0 \le y \le 3$ or [0, 3].

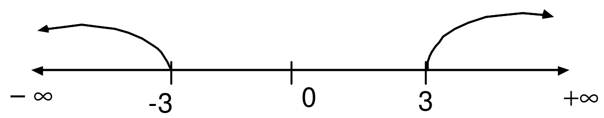
Example 2:

Find the domain and range of $y = \sqrt{x^2-9}$
Solution:
For domain,

the expression $\sqrt{x^2-9}$ is well defined if

$x^2 - 9 \ge 0$
or $x^2 \ge 9$
or $|x| \ge 3$
or $-3 \ge x$ and $x \ge 3$

This can be visualized in the diagram below



Hence domain is $(-\infty, -3] \cup [3, \infty)$.
Note that ∞ and - ∞ are not included in the real number system of this level. So always we put the open bracket for ∞ or - ∞.

For range,

$y = \sqrt{x^2-9}$
or $y^2 = x^2 - 9$
or $x = \sqrt{9+y^2}$

Here $\sqrt{9+y^2}$ is well defined for all values of y because for any value of y, the result under the root sign is always positive. Hence y can by any real number.
But the given function contains the positive square root. So the range contains only positive numbers.
Hence the range is $0 \le y < \infty$ or $[0, \infty)$.

B. Class work

Find the domain and range for the following functions:
i) $y = \sqrt{25-x^2}$ ii) $y = \sqrt{x^2-25}$ iii) $y = \sqrt{25+x^2}$

Lecture – 20 (50 minutes)

a) Types of function.
b) Inverse and Composite function.
c) Class work.

A. Types of function

Mainly injective, surjective and bijective functions are the types of functions. They can be defined as follows.

One to one (or injective) function:

Let f be a function from A to B. Then f is called a one to one function, if no two different elements in A have the same image in B.

In brief, $f: A \to B$ is one to one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Or $f: A \to B$ is one to one if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.

Examples:

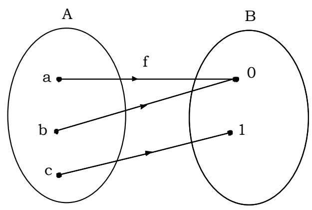
1. Let $f: \mathbf{R} \to \mathbf{R}$ be defined by the formula $f(x) = x^3$. Then f is one to one function since the cubes of different real numbers are different.
2. Let $f: \mathbf{N} \to \mathbf{R}$ be defined by the formula $f(x) = x^2$. Here the domain is the set of natural numbers. So, the function f is one to one since the squares of different natural numbers are different.
3. Let $f: \mathbf{R} \to \mathbf{R}$ be defined by the formula $f(x) = x^2$. Then f is **not** a one to one function because the image of two real numbers, 2 and -2 is the same number 4.

Onto (or surjective) function:

Let f be a function from A to B. Then f is said to be an onto function if every member of B appears as the image of at least one member of A. In this case $f(A) = B$. i. e. Co-domain and range of f are equal.

Examples:

1. Let A = {a, b, c} and B = { 1, 0}. Suppose $f: A \to B$ is defined by

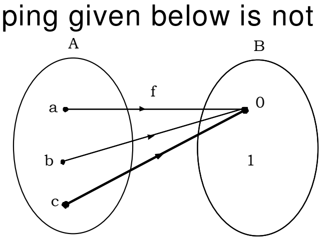


Then f is an onto function.

Here f(a) = f(b) = 0 and f(c) = 1,

But,

If A {a, b, c} and B = {1, 0}, $f: A \to B$ defined by the mapping given below is not an onto function.



Because 1 is not an image of any element of A.

Bijective (both injective and surjective) function:

Let f be a function from A to B. Then f is said to be a bijective function if it is both injective (one to one) and surjective (onto).

Examples:

1. Let A = { a, b, c} and B = {0, 1, 2}.
   Suppose f : A $\rightarrow$B is defined by
   f(a) = 0, f(b) = 1 and f(c) = 2,

Then f is a one to one function and also onto function. So, f is bijective function.

But,

2. For A= {a, b, c} and B= {1, 0}, f: A$\rightarrow$B
   defined by f(a) = 1, f(b) = 1, f(c) = 0

   is onto function but not one to one because two different elements a and b have the same image 0 in B. So, it is not bijective it is only onto function.

Also,

3. For A = {a, c} and B = {0, 1, 2}, f A$\rightarrow$B defined by

   f(a) = 0, f(c) = 1
   is an one to one function but not onto function because 2 is not an image of any element of A. So f is not a bijective function.

B. Inverse and Composite Function.

Inverse of a function:

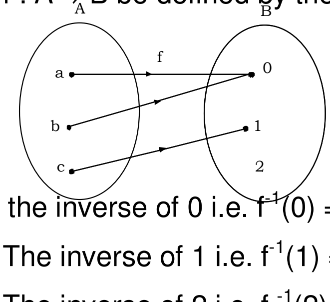
Let f be a function from A to B, and y $\in$B. Then the inverse of y, denoted by $f^{-1}(y)$, is the set of elements of A which have y as their image.

More precisely, if f: A$\rightarrow$ B, then

$$f^{-1}(y) = \{x : x \in A,\ y = f(x)\}.$$

Example:

Let f : A$\rightarrow$ B be defined by the arrow diagram



Then, the inverse of 0 i.e. $f^{-1}(0)$ = {a, b}

The inverse of 1 i.e. $f^{-1}(1)$ = {c}

The inverse of 2 i.e. $f^{-1}(2)$ does not exist

or $f^{-1}(2) = \phi$

Inverse function:

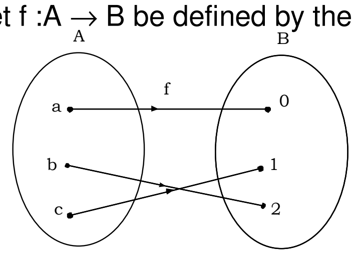
Let f be a function defined from A to B and $y \in B$. Then $f^{-1}(y)$ may consist of more than one element or one element or no element belonging to A. If f is bijective (one to one and onto) function then for each $y \in B$, the inverse $f^{-1}(y)$ consists of only one element of A. In such case, we can assign to each element y of B a unique element $f^{-1}(y)$ of A. So, f^{-1} is again a function from B to A. We write this as

$f^{-1}: B \to A.$

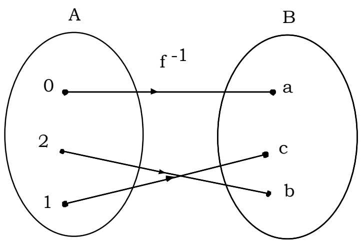
In short, if $f: A \to B$ is a bijective, then f^{-1} is well defined function and is known as the inverse function of f otherwise f^{-1} is not a function, it is just a relation.

Examples:

1. Let $f : A \to B$ be defined by the arrow diagram



Then the inverse function f^{-1} is defined by the arrow diagram



2. Let $f : \mathbf{R} \to \mathbf{R}$ and the real numbers be defined by $f(x) = x^3$. Then it is one to one and onto.
   Hence f^{-1} is defined. In fact the inverse function
   f may be defined by $f^{-1}(x) = \sqrt[3]{x}$

   <u>To find the rule or formula for $f^{-1}(x)$, we follow the following steps:</u>

   Here, $y = f(x) = x^3$. .... (1) Let f^{-1} be the inverse function so that

   $f^{-1}(y) = x$ ....(ii)

   Solving (1) for x in terms of y,

   $x^3 = y$

   Or $x = \sqrt[3]{y}$

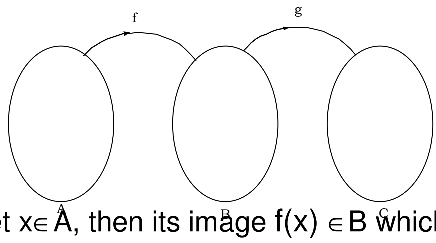
   Therefore from (ii) $f^{-1}(y) = \sqrt[3]{y}$

Hence $f^{-1}(x) = \sqrt[3]{x}$

[Note that y is merely a dummy variable and may be replaced by x]

Composite function:

Let $f : A \rightarrow B$ and $g: B \rightarrow C$ be any two functions. We may represent the functions by the following diagram:

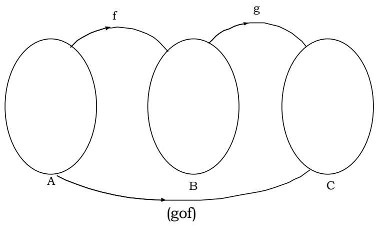


Let $x \in A$, then its image $f(x) \in B$ which is the domain of g. Accordingly, we can find the image of f(x) under g, i.e. we can find g(f(x)). Thus, we gave a rule which associates an element x of A with exactly one element of C. In other words, we have a new function from A to C. This new function is called the composite function of f and g **(not g and f),** and it is denoted by $(g_o f)$ or (gf).

More briefly, if $f : A \rightarrow B$ and $g : B \rightarrow C$, then

$(gf): A \rightarrow C$ is defined by $(g_o f)(x) = g(f(x))$.

Schematically the following diagram may illustrate the situation:



Examples:

1. Let f: **R** → **R** be defined by f(x) =3x and
g: **R**→ **R** by g(x) = 2x + 1. Then

$(g_o f)$**:R→R** is given by

$(g_o f)(x) = g(3x)$
$= 2(3x) + 1$ [since g(x)=2x+1]
$= 6x + 1$.

And $(f_o g)$: **R**→ **R** is given by

$(f_o g)(x) = f(g(x))$
$= f(2x + 1)$
$= 3(2x + 1)$ [since f(x) = 3x]
$= 6x + 3$

C. Class Work

1. Find the inverse of the following functions:

i) $f(x) = 3x - 2$ ii) $f(x) = 2x^2 + 3$. iii) $f(x) = {}^{(2x-1)}/_{(3x + 2)}$

2. Find the following composite functions if f(x) = 2x -3 and g(x)= 4x + 5

i) fg(x) ii) gfg(x) iii) $g^{-1}f(x)$

Lecture – 21 (50 minutes)

a) Applications of functions in business and economics.
b) Revenue and Profit functions.
c) Class works

A. Application of function in business and economics

many economic laws can be represented by a linear function. In general, demand is given by Q or Q_d = a – b P, where Q is the quantity demanded and P is the price and and b are constants. The supply is given by Q_s or simply Q = c + d.P, where Q is the quantity offered for sale and P is the price, c and d are constants.

Construction of a Function:

A simple cost function for a business consists of two parts: the fixed cost and variable cost. Fixed cost comprises rent, insurance, and business loans, which must be paid no matter how many items of a product are produced. Variable costs are labour cost, electricity charges etc. which depend on the number of items produced.

i.e. Total cost = fixed cost + variable cost.

Example:

Suppose a toy manufacturer has fixed costs of Rs 3000 (such as rent, insurance, and business loans) that must be paid no matter how many toys are produced. In addition, there are variable costs of Rs 2 per toy. Form a cost function and answer the following questions.

a) Find the cost of producing 2000 toys.
b) What additional cost is incurred if the production level is raised from 2000 toys to 2200 toys?
c) How many toys may be produced at a cost of Rs 5000?

Solution:

At a production level of x toys, the variable costs are 2x (rupees) and total cost is

C(x) = 3000 + 2x (rupees). [because 3000 is fixed cost and 2x is variable cost]

a) C(x) = 3000 + 2x

   C(2000) = 3000 + 2 (2000)

   = 7000 (rupees).

b) The total cost when x = 2200 is

   C(2200) = 3000 + 2 (2200) = 7400 (rupees)

So the increment in cost when production is raised from 2000 to 2200 toys is

C(2200) - C(2000) = 7400 - 7000

= 400 (rupees).

c) The phrase "how many toys" implies that the quantity x is unknown. Therefore, the answer is found by solving C(x) = 5000 for x. That is, C(x) = 5000

Or 3000 + 2x = 5000
Or 2x = 5000 - '11000 = 2000
$\therefore$ x = 1000.
Hence, 1000 toys may be produced by
Rs 5000.

B. Revenue and Profit functions

If p represents selling price per unit and q represents the quantity sold, then the total revenue is given by

revenue(R) = price per unit(p)×quantity sold(q)

Let C represent the total cost of production of q units of the product. The profit on the product is the difference between the amount of received by selling (i.e. revenue) and the cost invested for production (C). let π represent the profit. Then

Profit(π) = R – C.

Example:

Suppose a toy manufacturer has fixed costs of Rs 3000 (such as rent, insurance, and business loans) that must be paid no matter how many toys are produced. In addition, there are variable costs of Rs 2 per toy. Suppose the toys are sold for Rs 10 a piece. When x toys are sold, construct revenue function, profit function and answer the following questions.

a) Determine the revenue generated by 8000 toys.
b) If the revenue from the production and sale of some toys is Rs 7000, what is the corresponding profit?

Solution:

The revenue (amount of money received) R(x) is 10x rupees. Given the same cost function (as discussed above) C(x) = 3000 + 2x, the profit (or loss) π(x) generated by the x toys will be

$$\pi(x) = R(x) - C(x) = 10x - (3000 + 2x)$$
$$= 8x - 3000.$$

a) The revenue is unknown, but the input to the revenue function is known. So we have to compute R(8000) to find the revenue. We have, R(x) = 10x.
Hence R(8000) = 10 × 8000 = 80,000rupees.

b) The profit is unknown, so we want to compute the value of π(x). Unfortunately, we don't know the value of x. However, the fact that the revenue is Rs 7000 enables us to solve for x.

Thus the solution has two steps:
i) Solve R(x) =7000 to find x.
That is R(x) = 7000
Or 10x =7000

$\therefore$ x = 700.

ii) Compute $\pi(x)$ when x = 700.

That is,

$\pi(x) = 8x - 3000$

Or $\pi(x) = 8\,(700) - 3000$

$\therefore$ Profit = 2600 rupees.

C. Class work.

Solve the following problems:

1. In manufacturing a product, a firm incurs costs of two types. Fixed annual cost of Rs 100,000 is incurred regardless of the number of units produced. In addition, each unit produced costs of firm Rs. 5 equals total cost in rupees and x equals the number of units produced during a year.

   a) Determine the cost function C = f(x).
   b) What is C(20000)

2. The demand for a certain item is given by q = 150 - 3p here q is the amount demanded in unit and p, the price per unit. It costs Rs. 4 to produce each unit. What is revenue function? What is the profit function of the firm for this item? At what minimum price the firm should sell the item to receive a profit of Rs. 720?

3. A taxi charges Rs. 1.50 at the time of starting and Rs. 0.80 for each additional kilometer. If y be the total charge and x, the number of kilometers travelled, then determine the relationship y = f(x). Also find f(12).