Tenth week lessons

Quadratic Function

(Divided into 3 lectures of 50 minutes each)

Lecture – 28 (50 minutes)

a) Maximum and minimum values of a quadratic function.
b) Graphical representation of quadratic functions.
c) Class works

A. Maximum and minimum values of a quadratic function.

A function which has the general form $y = f(x) = ax^2 + bx + c$ is called the quadratic function, where a, b and c are real numbers, and $a \neq 0$.

The graph of a quadratic function is a parabola. If a>0, the parabola is concave up and if a<0 then the parabola is concave down. The value of y at the vertex is maximum if the curve is concave downward and the value of y at the vertex is minimum if the curve is concave up.

The vertex and the maximum or minimum value of a quadratic function can be obtained by completing the square. After completing the square a quadratic function looks like $y = f(x) =a(x+b)^2 + c$. Then the vertex is at (-b, c) and the maximum (if a is negative) value is c. The minimum value (for 'a' positive) is c.

Let us make it clear by an example.

Example:

For the quadratic equation $f(x) = 2x^2 + 6x + 5$, find the vertex and maximum or minimum value.

Solution:

$f(x) = 2x^2 + 6x + 5$
$=2(x^2 + 3x + 5/2)$
$= 2\{x^2 + 2.x.3/2 + (3/2)^2 + 5 – (3/2)^2\}$
$= 2\{(x + 3/2)^2 + 5 – 9/4\}$
$= 2\{(x + 3/2)^2 + 11/4\}$
$= 2. (x + 3/2)^2 + 11/2$

hence the vertex is (- 3/2, 11/2)

since coefficient of x^2 is 2>0; the curve is concave upwards. So, it gives minimum value. Minimum value is f(x) = 11/2 when x = - 3/2

Note: the vertex can be obtained by calculating $x = \frac{-b}{2a}$ and finding the corresponding f(x). i.e. vertex is $(\frac{-b}{2a}, f(x))$.

B. Graphical representation of quadratic functions.

After converting the quadratic equation into the square completed form : $f(x) = a(x+b)^2 + c$, we can note the following points.

i) the vertex is (-b, c)
ii) the curve is concave up or down for a +ve and –ve respectively.
iii) If a is +ve, the minimum value is c
iv) If a is –ve the maximum value is c
v) the line of symmetry of the parabolic graph is $x+b = 0$, or $x = -b$.

With these information, we can draw the graph of a quadratic function. This will be clear in the following example.

Example:

Draw the graph of $y = f(x) = x^2 + 4x + 3$

Solution:

First we complete the square of the given function.

$y = f(x) = x^2 + 4x + 3$
$= x^2 + 2.x.2 + (2)^2 + 3 - (2)^2.$
$= (x + 2)^2 - 1.$

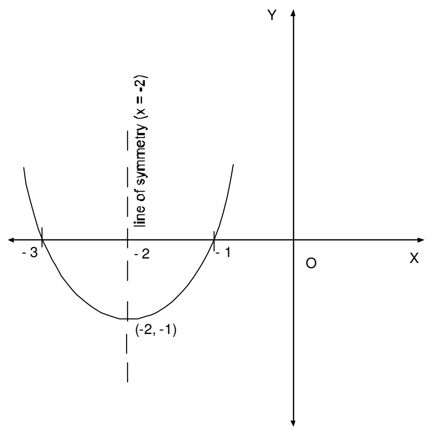
Hence we note that

i) the vertex is at (-2, -1)
ii) the curve is concave up because the coefficient of x^2 is +ve.
iii) the minimum value of the function is – 1.
iv) the line of symmetry of the parabolic graph is $x + 2 = 0$, or $x = -2$.

Further we can take some values of x and corresponding values of y to find some passing points.

i.e. if $x = 1$, $y = 8$, so (1, 8) is a passing point. Similarly, when $x = -1$, $y = 0$; when $x = -3$, $y = 0$. So, (-1, 0) and (-3, 0) are other two passing points.

So the graph looks like the following drawing.



C. Class works.

1. For the quadratic function $f(x) = x^2 - 6x + 14$, find
   i) the vertex.
   ii) concavity of the graph.
   iii) maximum or minimum value of the function.

2. Find the line of symmetry and vertex and the maximum or minimum value of the function $f(x) = 2x^2 + 5x + 15$
3. Sketch the graph of the following functions.
   i) $y = 3x^2 + 6x + 5$
   ii) $y = x^2 + 5$
   iii) $y = 4x^2$.

Lecture – 29 (50 minutes)

a) Revenue function, maximum and minimum revenue.
b) Profit function, maximum and minimum profit.
c) Class works

A. Revenue function, maximum and minimum revenue.

Revenue is the amount of money collected by selling Q units in price P each. So revenue (R) = P.Q. If Q is a linear function the revenue becomes a quadratic function in P. So we can find the maximum or minimum revenue and the corresponding price. The following example may clear the concept.

Example:

The demand function for a particular product is q = f(p) = 60,000 – 2,500p where q is stated in units and p is stated in dollars. Determine the quadratic total revenue function where R is a function of p, i.e. R = f(p).

i) What is the concavity of the revenue function?
ii) What is the total revenue when price is Rs 20?
iii) What is the maximum revenue?

Solution:

We know that revenue (R) = p.q
= p.(60,000 – 2,500p)
$= 60000p - 2500p^2$.

This is the required quadratic revenue function.

i) Since the coefficient of p^2 is negative; the graph is concave downwards.

ii) For p = 20, the revenue (R) $= 60000.\ 20 - 2500.\ (20)^2$.
= 1200000 – 1000000
= 200000.

iii) For maximum revenue, we write the revenue function in the square completed form.

i.e. $R = 60000p - 2500p^2$.
$= -2500p^2 - 60000p$
$= -2500\ (p^2 + 24p)$
$= -2500\ (p^2 + 2.p.\ 12 + 12^2 - 12^2)$
$= -2500\ \{(p + 12)^2 - 144\}$
$= -2500(p + 12)^2 + 36000$

Hence the maximum revenue is 36000.

B. Profit function, maximum and minimum profit.

Profit is the difference of selling price (revenue) and cost. If profit is denoted by π, then $\pi = R - C$. Since revenue function is generally a quadratic function, the profit function is also in quadratic form in general. So, we follow the similar process for profit maximization as we did in revenue maximization. Let us make it clear by an example.

Example:
The demand for a product is q = 7000 – 50p, where p is the price. The cost function is given by 2000 + 200 p. Form a revenue function and hence find the profit function. Then calculate the followings.

i) concavity of the profit function.
ii)maximum profit.

Solution:
we know that revenue (R) = p.q

$$= p.(7000 - 50p)$$
$$= 7000p - 50p^2.$$

we have cost (C) = 2000 + 200 p.

So, profit (π) = R – C

$$= 7000p - 50p^2 - 2000 - 200p.$$
$$= -50\,p^2 - 6800\,p - 2000.$$

i) since the coefficient of p^2 is negative, its graph is concave downwards.
ii) For maximum profit, we complete the square.
That is,

$$\pi = -50\,(p^2 - 136\,p - 40)$$
$$= -50\,(p^2 - 2.p.\,68 + 68^2 - 68^2 - 40)$$
$$= -50\,\{(p - 68)^2 - 4624 - 40\}$$
$$= -50\,\{(p - 68)^2 - 4664\}$$
$$= -50(p - 68)^2 + 233200$$

Hence the maximum profit is Rs 233200.

C. Class Works

1. If the total revenue function (R) = $-3x^2 + 900x + 1200$, find the maximum revenue.
2. If the cost function C = 20x + 50 and revenue function R = $-5x^2 + 60x$, form a profit function and calculate the maximum profit.

Lecture – 30 (50 minutes)

a) market equilibrium
b) break even analysis.
c) Class works

A. Market equilibrium.

The demand function gives the price in terms of quantity demanded and the supply function gives the price in terms of the quantity supplied. The point at which the price of demand and that of supply are equal, the demanded quantities and the supplied quantities are also equal. That point is known as the point of equilibrium. The market equilibrium gives the equilibrium price and quantity in the market.

For example:
If the demand function for a commodity is given by $p = 4 - q^2$ and the supply function is given by $p = q + 2$, find the market equilibrium.
Solution:
here, the demand function is $p = 4 - q^2$ and supply function is $p = q + 2$.
we that at market equilibrium, quantity demanded = quantity supplied.

$$\begin{aligned} &\text{i.e. } 4 - q^2 = q + 2 \\ &\text{or } q^2 + q - 2 = 0 \\ &\text{or } q^2 + 2q - q - 2 = 0 \\ &\text{or } q(q + 2) - 1(q + 2) = 0 \\ &\text{or } (q + 2)(q - 1) = 0 \\ &\text{so, } q = 1 \text{ or } -2. \end{aligned}$$

Since q cannot be negative, we take $q = 1$ only. When $q = 1$, $p = 3$. So the point of market equilibrium is (1, 3).

B. Break Even Analysis.

When the profit and revenue of a company are equal, we say the company is in break even condition. In this condition, the company bears neither loss nor profit. The company sustains without any loss and profit. We can analyze the values of price and demand at the break even point. Mathematically we take revenue(R) = cost(C) to find the break even point. Let us see an example.

- The weekly cost of manufacturing x telephones per week is found by a manufacturer to be $C(x) = 500 + 20x + x^2$ dollars. The telephone can be sold at a price of p = $80 each.
  a) Find the manufacturer's break even point of production levels.
  b) For what production level will the manufacturer experience a profit?

Solution:
We have cost function $C(x) = 500 + 20x + x^2$.
revenue (R) = p. q = 80.x
a) For break even point, we take R = C

$$\begin{aligned} &\text{or } 80x = 500 + 20x + x^2. \\ &\text{or } x^2 - 60x + 500 = 0 \\ &\text{or } x^2 - 50x - 10x + 500 = 0 \\ &\text{or } x(x - 50) - 10(x - 50) = 0. \\ &\text{or } (x - 50)(x - 10) = 0 \\ &\text{or } x = 50,\ x = 10. \end{aligned}$$

Hence the production level is 10 or 50 for the break even point.
b) To experience the profit, we must have R>C

$$\begin{aligned} &\text{or } 80x > 500 + 20x + x^2. \\ &\text{or } x^2 - 60x + 500 < 0 \\ &\text{or } x^2 - 50x - 10x + 500 < 0 \\ &\text{or } x(x - 50) - 10(x - 50) < 0. \\ &\text{or } (x - 50)(x - 10) < 0 \\ &\text{or } 10 < x < 50. \end{aligned}$$

[how?, see the solution of quadratic inequalities discussed in previous lesson.]

C. Class works

1. If the supply function for a commodity is $p = x^2 + 8x + 20$ and the demand function is $p = -x^2 + 4x + 100$, find the equilibrium quantity and equilibrium price.
2. If total costs are $C(x) = 15000 + 35x + 0.1\, x^2$ and total revenues are $R(x) = 385x - 0.9\, x^2$, find the break even points.