Twelfth week lessons

Matrices and Determinants

(Divided into 3 lectures of 50 minutes each)

Lecture – 34 (50 minutes)

a) Exercise of the problems related to matrix
b) Assignment

A. Exercise of the problems related to Matrix

1. Construct a 3×2 matrix whose elements aij are given by aij = j - 2i.

2. If A = $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, show that $A^2 - 4A - 5I = 0$, where I is a 3×3 unit matrix.

3. Write the sizes of the following matrices.

i) $\begin{bmatrix} 3 & 2 & 5 \\ 4 & -1 & 0 \\ 6 & 7 & 8 \end{bmatrix}$ ii) $\begin{bmatrix} 3 & 1 & 2 \\ 4 & 1 & 6 \\ 5 & 3 & 0 \end{bmatrix}$ iii) $\begin{bmatrix} 5 & 1 \\ 7 & 3 \end{bmatrix}$

iv) $\begin{bmatrix} 4 & 3 & 4 \\ 2 & 9 & 2 \\ 1 & 5 & 1 \end{bmatrix}$ v) $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$

4. Are matrix multiplications always possible?

If A = $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and B = $\begin{bmatrix} 3 & 1 & 2 \\ 4 & 5 & 3 \\ 5 & 1 & 4 \end{bmatrix}$, are BA and AB both possible? Give reason. Find A.B or B.A whichever is possible.

5. If A = $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 4 & 1 \\ 4 & 2 & 3 \end{bmatrix}$, Find 2A - 3B.

B. Assignment.

1. Suppose your mathematics teacher took three term tests of your class in order to award internal marks. Your teacher decided to weight the first test at 20%, second at 30% and the third at 50%. The matrix of marks (out of 100) you scored is M = [91 95 100]. Apply matrix multiplication to compute your internal marks (out of 100) in mathematics.

2. If A = $\begin{bmatrix} 3 & 2 \\ -1 & 6 \end{bmatrix}$ and B = $\begin{bmatrix} 6 & -2 \\ 1 & 3 \end{bmatrix}$, find AB and BA.

3. Find the value of $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \end{bmatrix}$

4. If A = $\begin{bmatrix} 2m & 7 \\ 5 & 9 \end{bmatrix}$ and B = $\begin{bmatrix} 9 & n \\ -5 & 4 \end{bmatrix}$ and A.B = I, where I is and identity matrix of size 2, find the values of m and n.

5. There are four book shops A, B, C and D. A has 82 books on Algebra, 64 books on Geometry and 78 books on Economics; B has 75 books on Algebra, 38 books on Geometry and 85 books on Economics; C has 56 books on Algebra and 106 books on Economics; D has 94 books on Algebra and 74 books on Geometry. Use vector addition to find the total number of each kind of books.

6. Three shops, P, Q and R sell three items X, Y and Z in the three shops. The following matrix A shows the stock of X, Y and Z in the three shops. Matrix B shows the number of each item received at the beginning of the week. Matrix C shows the sales during the week.

$$A = \begin{array}{c} \\ X \\ Y \\ Z \end{array}\begin{array}{c} \begin{array}{ccc} P & Q & R \end{array} \\ \begin{bmatrix} 4 & 5 & 6 \\ 5 & 6 & 7 \\ 6 & 7 & 8 \end{bmatrix} \end{array}, \quad B = \begin{array}{c} \\ X \\ Y \\ Z \end{array}\begin{array}{c} \begin{array}{ccc} P & Q & R \end{array} \\ \begin{bmatrix} 3 & 4 & 5 \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix} \end{array}, \quad C = \begin{array}{c} \\ X \\ Y \\ Z \end{array}\begin{array}{c} \begin{array}{ccc} P & Q & R \end{array} \\ \begin{bmatrix} 3 & 3 & 4 \\ 3 & 4 & 4 \\ 4 & 4 & 5 \end{bmatrix} \end{array}$$

Find (a) the number of items after receiving the goods,

(b) the number of items at the end of the week,

(c) the number of items to be ordered so that the stock of all items in all shops will be 8.

Lecture – 35 (50 minutes) Determinant.

a) Determinant (introduction)
b) Minors, cofactors and determinants.
c) Class works

A. Determinants (Introduction).

The determinant is a single number or scalar and is found only for square matrices. If the determinant of a matrix is equal to zero, it is said to be singular. A singular matrix is one in which there exists linear dependence between at least two rows or columns. If $|A| \neq 0$, matrix A is nonsingular and all its rows and columns are linearly independent.

B. Minor, cofactors and determinants.

i) Minor

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a square matrix of order 3.

By deleting the first row and first column of this matrix, we get $\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$

The determinant of this matrix is $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ which is called the minor of element a_{11}. generally it is denoted by M_{11}.

Similarly $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$, $M_{23} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$ and so on. Each element of a square matrix has its minor.

ii) Cofactor

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a square matrix of order 3.

The cofactor of a_{11} is $(-1)^{1+1}$ times the minor of a_{11}. It is denoted by A_{ij}. So in general $A_{ij} = (-1)^{i+j} \times M_{ij}$.

iii) Determinants.

Determinant of a square is calculate by expanding from a row or column of the matrix. The sum of the products of elements of only one row and their respective cofactors gives the value of determinant. The determinant of A is denoted by $|A|$.

i.e. The determinant of 2×2 matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}.a_{22} - a_{21}.a_{12}$$

And the determinant of 3×3 matrix $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}.A_{11} + a_{12}.A_{12} + a_{13}.A_{13}.$$ [here A_{11}, A_{12}, A_{13} are cofactors]

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}.[a_{22}.a_{33}-a_{32}.a_{23}] - a_{12}.[a_{21}.a_{33} - a_{31}.a_{23}]$$

$$+ a_{13}[a_{21}.a_{32} - a_{31}.a_{22}]$$

C. **Class works.**

1. Evaluate $\begin{vmatrix} 2 & 1 & 1 \\ 5 & 6 & 3 \\ 4 & 7 & 5 \end{vmatrix}$

2. Write the minors and cofactors of a_{21}, a_{22}, a_{23} for $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

Lecture – 36 (50 minutes).

a) Properties of determinants with examples.
b) Class works

A. Properties of determinants with examples.

1. The value of the determinant remain unaltered by changing its rows and columns.
   Example:
   $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

2. If any two adjacent rows (or columns) are interchanged, the sign of the determinant will be changed.
   Example:
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix}$$

3. If any two rows (or two columns) of a determinant are identical, then the value of determinant is zero.
   Example:
$$\begin{vmatrix} a_1 & b_1 & b_1 \\ a_2 & b_2 & b_2 \\ a_3 & b_3 & b_3 \end{vmatrix} = 0$$

4. If all the elements of any one row (or column) are zero, then the value of determinant is zero.
   Example:
$$\begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} = 0$$

5. If a multiple of a row (or column) is added to or subtracted from any other row (or column) then the value of determinant will be unaltered. [note that the multiple can be 1 or any real number]
   Example:
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + k.a_2 & b_1 + k.b_2 & c_1 + k.c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

6. We can multiply to or take common from one row (or column). If we multiply inside, we need to balance by dividing outside.
   Example:
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \frac{1}{k} \begin{vmatrix} a_1 & k.b_1 & c_1 \\ a_2 & k.b_2 & c_2 \\ a_3 & k.b_3 & c_3 \end{vmatrix}$$

   We use the above stated rules to prove some results of determinants. Let us see an example.

Example 1:

Prove: $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a - b)(b - c)(c - a)$

Solution:

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$= \begin{vmatrix} 1 & 0 & 1 \\ a & b-a & c \\ a^2 & b^2-a^2 & c^2 \end{vmatrix}$ by $C_2 \to C_2 - C_1$ [property 5]

$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$ by $C_3 \to C_3 - C_1$

$= \begin{vmatrix} b-a & c-a \\ b^2-a^2 & c^2-a^2 \end{vmatrix}$ expanding along R_1

$=(b-a)(c-a)\begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix}$ taking b-a & c-a common from C_1&C_2

$= (b - a)(c - a)(c + a - b - a)$

$= (b - a)(c - a)(c - b)$

$= (a - b)(b - c)(c - a)$ Proved.

Example 2:

Prove that $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a + b + c)^3$.

Solution:

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$$

$$= \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix} \text{ by } C_1 \to C_1 + C_2 + C_3$$

$$= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix} \text{ taking } 2(a+b+c) \text{ common from } C_1$$

$$=2(a+b+c) \begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 1 & a & c+a+2b \end{vmatrix} \text{ By } R_2 \to R_2 - R_1$$

$$=2(a+b+c) \begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & c+a+b \end{vmatrix} \text{ By } R_3 \to R_3 - R_1$$

$$= 2(a+b+c) \begin{vmatrix} b+c+a & 0 \\ 0 & c+a+b \end{vmatrix} \text{ expanding along } C_1$$

$= 2(a+b+c)\ [(a+b+c)\ (a+b+c) - 0]$

$= 2(a+b+c)^3$.

B. Class Works.

Prove the following identities.

a) $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

b) $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (a - b)(b - c)(c - a).$

c) $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = abc(a - b)(b - c)(c - a).$

d) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a + b + c)(a - b)(b - c)(c - a).$