Thirteenth week lessons

Determinants

(Divided into 3 lectures of 50 minutes each)

Lecture – 37 (50 minutes)

a) Adjoint and Inverse of a matrix.
b) Class works.

A. Ajoint and inverse of a matrix

Adjoint of a matrix is the transpose of the matrix of cofactors. So, to find adjoint of a matrix A, we find the cofactors of each element and make a new matrix of cofactors. The transpose of the matrix of cofactors is known as the adjoint of the given matrix A. Note that the transpose of a matrix is obtained by interchanging the rows and columns. We will see an example to find adjoint of a matrix while finding the inverse.

An inverse matrix A^{-1}, which can be found only for a square, non singular matrix A, is a unique matrix satisfying the relationship

$$AA^{-1} = I = A^{-1}A$$

Here I is the identity matrix of same size. Note that a matrix whose determinant is zero is known as a singular matrix.

The formula for deriving the inverse is

$$A^{-1} = \frac{1}{|A|}.\mathrm{adj}(A)$$

where |A| is determinant of A and adj(A) is the adjoint of A. The adjoint of A is obtained by the transpose of the matrix of cofactors as explained above.

For Example:

Find the inverse of $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$.

Solution:

$$|A| = 1\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= 1(2 - 2) - 2(6 - 2) + 1(3 - 1)$$

$$= 0 - 8 + 2$$

$= -6 \neq 0$, so inverse exists.

Now, finding the cofactors of each element:

$A_{11} = +\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0$

$A_{12} = -\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} = -4$

$A_{13} = +\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = 2$

$A_{21} = -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = -3$

$A_{22} = +\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$

$A_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1$

$A_{31} = +\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$

$A_{32} = -\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = 1$

$A_{33} = +\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5$

$\therefore$ matrix of cofactors is $\begin{pmatrix} 0 & -4 & 2 \\ -3 & 1 & 1 \\ 3 & 1 & -5 \end{pmatrix}$

$\therefore$ adjoint of A = Transpose of $\begin{pmatrix} 0 & -4 & 2 \\ -3 & 1 & 1 \\ 3 & 1 & -5 \end{pmatrix}$

$$= \begin{pmatrix} 0 & -3 & 3 \\ -4 & 1 & 1 \\ 2 & 1 & -5 \end{pmatrix}$$

Hence $A^{-1} = \frac{1}{|A|}.\text{Adj}(A)$

$$= \frac{1}{-6}\begin{pmatrix} 0 & -3 & 3 \\ -4 & 1 & 1 \\ 2 & 1 & -5 \end{pmatrix}.$$

B. Class works

1. What do you mean by a singular matrix?
2. Find the adjoint of A, if A = $\begin{pmatrix} 0 & 3 & 3 \\ 4 & 1 & 1 \\ 2 & 1 & 5 \end{pmatrix}$
3. Find B^{-1} if B = $\begin{pmatrix} 0 & 2 & 0 \\ 4 & 1 & 1 \\ 1 & 2 & 5 \end{pmatrix}$

Lecture – 38 (50 minutes)

a) Cramer's rule to solve the system of linear equations
b) Class works.

A. **Cramer's rule to solve the system of linear equations**

This rule provides a simplified method of solving a system of linear equations through the use of determinants.

Cramer's rule states that $x_i = \frac{|A_i|}{|A|}$

where, xi is the ith unknown variable in a series of equations, |A| is the determinant of the coefficient matrix, and |Ai| is the determinant of a special matrix formed from the original coefficient matrix by replacing the column of coefficients of xi with the column vector of constants.

Example

Use Cramer's rule to solve the system of equation

2x + 3y = 13 and 3x - y = 3

Solution

[Coeff. of x]	[Coeff. of y]	[Constant]
2	3	13
3	-1	3

D (determinant of coefficients of variables) $= \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix}$

= -2 - 9

= -11

D1 (determinant obtained by replacing the

coefficients of x by constants) $= \begin{vmatrix} 13 & 3 \\ 3 & -1 \end{vmatrix}$

= -13 - 9

= -22

D2 (determinant obtained by replacing the

coefficients of y by constants) $= \begin{vmatrix} 2 & 13 \\ 3 & 3 \end{vmatrix}$

= 6 - 39

= -33

Hence by Cramer's rule $x = \frac{D_1}{D} = \frac{-22}{-11} = 2$

$y = \frac{D_2}{D} = \frac{-33}{-11} = 3$

Note : To solve the linear equations of three variables, we find the determinants of 3×3 size. There will be D, D_1, D_2, D_3. Then we use $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$

B. Class works

1. Solve the following systems of linear equations by Cramer's rule.

   i) $2x + 5y = 7$, $-3x + 2y = -1$

   ii) $x + y + z = 6$, $2x - y = 0$, $y + 3z = 11$

Lecture – 39 (50 minutes)

a) Inverse matrix method to solve the system of linear equations
b) Class works.

A. Inverse matrix method to solve the system of linear equations.

An inverse matrix can be used to solve matrix equations. If two equations are

$a_1x + b_1y = c_1$ and

$a_2x + b_2y = c_2$

their representation in matrix form is

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

If we suppose $A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $B = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

Then $A.X = B$

If $|A| \neq 0$, the equations are solvable because the inverse A^{-1} exists. Multiplication of both sides of the equation by A^{-1}, gives

$$X = A^{-1}.B$$

by which the values of x and y are obtained.

Example

Solve the system of linear equations

$$x + 2y + z = 8$$

$$3x + y + 2z = 11$$

$$x + y + 2z = 9$$

Solution

The given equations can be written in matrix form as:

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 11 \\ 9 \end{pmatrix}$$

Suppose $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B = \begin{pmatrix} 8 \\ 11 \\ 9 \end{pmatrix}$

Then

$$AX = B$$

Or $\quad X = A^{-1}B \ldots \ldots \ldots \ldots [1]$

Now we find the inverse of A. For this

$$|A| = 1\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= 1(2 - 2) - 2(6 - 2) + 1(3 - 1)$$

$$= 0 - 8 + 2$$

$= -6 \neq 0$, so inverse exists.

Now, finding the cofactors of each element:

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$\therefore$ matrix of cofactors is $\begin{pmatrix} 0 & -4 & 2 \\ -3 & 1 & 1 \\ 3 & 1 & -5 \end{pmatrix}$

$\therefore$ adjoint of A = Transpose of $\begin{pmatrix} 0 & -4 & 2 \\ -3 & 1 & 1 \\ 3 & 1 & -5 \end{pmatrix}$

$$= \begin{pmatrix} 0 & -3 & 3 \\ -4 & 1 & 1 \\ 2 & 1 & -5 \end{pmatrix}$$

Hence $A^{-1} = \frac{1}{|A|}.\mathrm{Adj}(A)$

$$= \frac{1}{-6}\begin{pmatrix} 0 & -3 & 3 \\ -4 & 1 & 1 \\ 2 & 1 & -5 \end{pmatrix}$$

Therefore from [1]

$$X = A^{-1}B$$

$$= \frac{1}{-6}\begin{pmatrix} 0 & -3 & 3 \\ -4 & 1 & 1 \\ 2 & 1 & -5 \end{pmatrix}.\begin{pmatrix} 8 \\ 11 \\ 9 \end{pmatrix}$$

$$= \frac{1}{-6}\begin{pmatrix} 0.8+(-3).11+3.9 \\ -4.8+1.11+1.9 \\ 2.8+1.11+(-5).9 \end{pmatrix} = \frac{1}{-6}\begin{pmatrix} -6 \\ -12 \\ -18 \end{pmatrix}$$

i.e. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{-6}\begin{pmatrix} -6 \\ -12 \\ -18 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Hence x = 1, y = 2 and z = 3.

B. Class works.

Solve the following systems of linear equations by inverse matrix method.

i) 2x + 5y = 7, -3x +2y = -1

ii) x + y + z = 6, 2x – y = 0, y + 3z = 11