Fourteenth week lessons

Determinants

(Divided into 3 lectures of 50 minutes each)

Lecture – 40 (50 minutes)

a) Business related problems of matrix and determinants.
b) Class works.

A. Business related problems of matrix and determinants.

The concept of matrix and determinant is used to display the results of business and economics. Some problems seem to be clear when they are solved using the concept of matrix and determinants. Let us see two examples.

Example 1:

The cost of manufacturing the three types of motor cars is given below:

Car	Labour hrs	Material used	Subcontracted work
A	40	100 units	50 units
B	80	150 units	80 units
C	100	250 units	100 units

The labour cost is Rs 20 per hour, per unit material cost is Rs 100 and one unit of subcontracted work costs Rs 10. Find the total cost of manufacturing 3000, 2000 and 1000 vehicles of types A, B and C respectively.

Solution:

Consider the following matrices

$$M = \begin{bmatrix} 40 & 100 & 50 \\ 80 & 150 & 80 \\ 100 & 250 & 100 \end{bmatrix} \text{ and } N = \begin{bmatrix} 20 \\ 100 \\ 10 \end{bmatrix}$$

Here, M represents labour hours, material used and subcontracted work for three types of cars A, B and C respectively.

The column matrix N represents labour cost per unit, material cost and the cost of subcontracted work.

So, the cost of manufacturing one of each types of cars A, B, C is given by the product M.N.

$$M.N = \begin{bmatrix} 40 & 100 & 50 \\ 80 & 150 & 80 \\ 100 & 250 & 100 \end{bmatrix} . \begin{bmatrix} 20 \\ 100 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} 800 + 10000 + 500 \\ 1600 + 15000 + 800 \\ 200 + 25000 + 1000 \end{bmatrix} = \begin{bmatrix} 11300 \\ 17400 \\ 28000 \end{bmatrix}$$

Again, to find the cost of producing 3000, 2000 and 1000 units of A, B and C we write the matrix $\begin{bmatrix} 3000 & 2000 & 1000 \end{bmatrix}$ so that the total cost is given by the product

$$\begin{bmatrix} 3000 & 2000 & 1000 \end{bmatrix} . \begin{bmatrix} 11300 \\ 17400 \\ 28000 \end{bmatrix}$$

= 33900000+34800000+28000000 = Rs. 96700000.

Example 2:
If p_1, p_2 be the prices of two commodities whose demand and supply functions are as follows:

$x_{d1} = 4 - 3p_1 + p_2 \qquad x_{s1} = -5 + p_1$

$x_{d2} = 9 + p_1 - 4p_2 \qquad x_{s2} = -6 + 2p_2$.

Find the equilibrium prices.

Solution:

For equilibrium prices, demand = supply

That is, $x_{d1} = x_{s1}$

or $4 - 3p_1 + p_2 = -5 + p_1$

or $9 = 4p_1 - p_2$ ... ... [1]

Also,

$x_{d2} = x_{s2}$

or $9 + p_1 - 4p_2 = -6 + 2p_2$

or $15 = -p_1 + 6p_2$ ... ... [2]

To solve equations [1] and [2] for p_1 and p_2, we use Cramer's rule.

For this we write the coefficients and constants as shown below.

coefficient of p_1	coefficient of p_2	constant
4	-1	9
-1	6	15

so, $D = \begin{vmatrix} 4 & -1 \\ -1 & 6 \end{vmatrix} = 24 - 1 = 23$

$D_1 = \begin{vmatrix} 9 & -1 \\ 15 & 6 \end{vmatrix} = 54 + 15 = 69$

and $D_2 = \begin{vmatrix} 4 & 9 \\ -1 & 15 \end{vmatrix} = 60 + 9 = 69$

So by Cramer's rule $p_1 = \frac{D_1}{D} = \frac{69}{23} = 3$ and $P_2 = \frac{D_2}{D} = \frac{69}{23} = 3$

Hence the equilibrium prices are 3 and 3.

B. Class Works.

1. If p_1, p_2 be the prices of two commodities whose demand and supply functions are as follows:

   $x_{d1} = 82 - 3p_1 + p_2$ $\quad x_{s1} = -5 + 15p_1$

   $x_{d2} = 92 + 2p_1 - 4p_2$ $\quad x_{s2} = -6 + 32p_2$. Find the equilibrium prices.

[Hint: for equilibrium prices, take $x_{d1} = d_{s1}$ and $x_{d2} = x_{s2}$ and making two equations solve them by Cramer's rule]

2. For the following demand and supply functions, find the equilibrium prices.

   $D_1 = 4p_1 - 3p_2 + 2,$ $\quad S_1 = 8$ and

   $D_2 = p_1 + 2p_2 + 5,$ $\quad S_2 = 12.$

3. A bus has four seats for passengers. Those willing to pay the first class fare can take 60 kg. of baggage each but tourist class passengers are restricted to 20 kg. each. If only 120 kg. of baggage can be carried together, find the no. of passengers of each kind by using determinant.

   [Hints: suppose the number of tourist passenger be x and that of first class be y. Then form two equations and solve them using Cramer's rule or Inverse matrix method.]

Lecture – 41 (50 minutes)

a) Rank of a matrix
b) Class works.

A. Rank of a matrix

A matrix A is said to be of rank r, if it possesses at least one non zero minor of order r and every minor of order r+1 is zero.

For example:

1. If A = $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 3 & 6 & 3 \end{bmatrix}$ then

   $|A| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 3 & 6 & 3 \end{vmatrix} = 0$, so rank is not 3

   And

   The given matrix A has 9 minors of order 2. In addition, one of them $\begin{vmatrix} 4 & 1 \\ 6 & 3 \end{vmatrix} = 12 - 6 = 6$

   which is not zero.

   Hence the rank of the given matrix is 2.

2. Find the rank of the matrix A = $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ -1 & -3 & -4 & -3 \end{bmatrix}$

   Solution:
   There are four minors of order 3 which are
   $\begin{bmatrix} 1 & 3 & 4 \\ 3 & 9 & 12 \\ -1 & -3 & -4 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 & 3 \\ 3 & 9 & 9 \\ -1 & -3 & -3 \end{bmatrix}$, $\begin{bmatrix} 1 & 4 & 3 \\ 3 & 12 & 9 \\ -1 & -4 & -3 \end{bmatrix}$ and
   $\begin{bmatrix} 3 & 4 & 3 \\ 9 & 12 & 9 \\ -3 & -4 & -3 \end{bmatrix}$
   All of the minors of order 3 have determinant zero. So rank is not 3.

Now we take the minors of order 2. There are nine minors of size 2 of each of the above written matrices. We can see that each of them has determinant zero. So rank is not 2.

Lastly, there are 12 minors of order 1. None of them has determinant zero. (note that only one non zero determinant is sufficient). So the Rank = 1.

B. Class works.

Find the rank of the following matrices.

i) $\begin{bmatrix} 2 & 2 & 1 \\ 4 & 4 & 4 \\ 3 & 6 & 3 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$ iii) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 4 \\ 3 & 2 & 3 \end{bmatrix}$

Lecture – 42 (50 minutes)

a) Assignment and its discussion.

1. Solve by Cramer's rule:

i) $x + 2y = 7, 2x - y = 4$ ii) $9x + 11y - 20 = 0, 5y + 6x - 11 = 0$

iii) $3x + 20 = 4y - 10, 4(x - 1) = 3(y - 3)$ iv) $\frac{2x}{3} + y = 16, x + \frac{y}{4} = 14$

v) $\frac{4}{x} + \frac{5}{y} = 58, \frac{7}{x} + \frac{3}{y} = 67$

[Hints: take 1/x and 1/y as the variables and take their coefficients. Then use Cramer's formula as $\frac{1}{x} = \frac{D_1}{D}$ and $\frac{1}{y} = \frac{D_2}{D}$, then find x and y]

vi) $x + y + z = 7, x + 2y + 3z = 16, x + 3y + 4z = 22$

vii) $y + 3x + z = 10, x + y - z = 0, 5x - 9y = 1$

viii) $2x + 2y + z = 16, x + 2y = 10, 2y + 3z = 12$

ix) $x_1 + 3x_2 - 2x_3 = 17, 2x_1 - 4x_2 + x_3 = -16, 5x_1 + 2x_2 - 4x_3 = 21$.

2. Find the adjoint of the matrix $A = \begin{bmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{bmatrix}$.

3. Given a matrix A = $\begin{bmatrix} 1 & 6 & -1 \\ 2 & 3 & 3 \\ 3 & 3 & -2 \end{bmatrix}$ (i) Find |A| (ii) Does A^{-1} exist? Why? (iii) If exists, find A^{-1}.

4. If A = $\begin{bmatrix} 1 & 4 & 3 \\ 3 & -3 & 1 \\ -1 & 1 & 2 \end{bmatrix}$. Find the determinant of A by cofactors A_{11}, A_{12}, A_{13}.

5. A company produces two commodities P and Q which must pass through machine M and N. One unit of P requires 12 minutes of work on machine M and 5 minutes of work on machine N. Similarly, one unit of Q requires 5 minutes of work on machine M and 12 minutes of work on machine N. How many units of P and Q are produced if M operates for 2 hours and N operates for 2 hours 49 minutes? Apply Cramer's rule.

6. The quantities of protein, vitamin and carbohydrate in each packet of Horlicks, Viva and Bournvita are given below:

	Protein	**Vitamins**	**Carbohydrates**
Horlicks	4	3	1
Viva	1	4	3
Bournvita	3	1	4

If the total quantities of protein, vitamin and carbohydrate taken by a patient in a month are 20, 19 and 17 units, find the number of packets of Horlicks, Viva and Bournvita used by the patient in the month by applying Cramer' rule.

7. A manufacturer of leather belts makes three types of belts A, B and C which are processed on three machines M, N and P. Belt A requires 2 hours on M and 1 hour on N. Belt B requires 2 hours on M, 2 hours on N and 2 hours on P. Belt C requires 1 hour on M and 3 hours on P. M, N and P are available daily for 16, 10 and 12 hours respectively. What should be the daily production of each of the belts so that the whole time is utilized?

8. There are three brands of fertilizers A, B and C. A contains 1 unit of nitrates, 2 units of potash and 3 units of phosphates. B contains 3 units of nitrates, 1 unit of potash and 2 units of phosphates. C contains 2 units of nitrates, 3 units of potash and 1 unit of phosphates. If 11 units of nitrates, 10 units of potash and 9 unit of phosphates are necessary for a field, how much of each type of fertilizer is required for it?

9. A, B and C have Rs.96, Rs. 150 and Rs. 142 respectively. They use their entire amount to purchase three types of articles of prices x, y and z respectively. A purchases 2 articles of price x, 5 of price y and 3 of price z. B purchases 4 articles of price x, 3 of price y and 6 of price z. C purchases 1 article of price x, 4 of price y and 10 of price z. Find x, y and z.

10. If A = $\begin{bmatrix} 3 & 2 \\ -1 & 6 \end{bmatrix}$ and B = $\begin{bmatrix} 6 & -2 \\ 1 & 3 \end{bmatrix}$, find AB and BA.

11. Find the value of $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}$

12. If A = $\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$, B = $\begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}$ and C = $\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$, find A(BC).

13. Find the Rank of i) $\begin{bmatrix} 1 & 4 & 3 \\ 3 & -3 & 1 \\ -1 & 1 & 2 \end{bmatrix}$ ii) $\begin{bmatrix} 5 & 2 \\ 4 & 3 \end{bmatrix}$ iii) $\begin{bmatrix} 1 & 4 & 3 \\ 3 & -3 & 1 \\ 1 & 4 & 3 \end{bmatrix}$

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