FOURIER SERIES AND PDEs
LECTURE 4

CONVERGENCE OF FOURIER SERIES

For the previous example it does appear that except at points of discontinuity the partial sums do converge to $f(x)$. At points of discontinuity they converge to zero. A similar result is true also for most functions which appear in applications. To present this result we first need to discuss one-sided limits.

Definition We say that the *right-hand* limit of f at c is

$$f(c_+) = \lim_{\substack{h\to 0\\ h>0}} f(c+h), \tag{2.51}$$

if this exists. Similarly, the *left-hand limit* of f at c is

$$f(c_-) = \lim_{\substack{h\to 0\\ h>0}} f(c-h), \tag{2.52}$$

if this exists.

The existence part is important since, for example, $f(x) = \sin(1/x)$ does not have these limits at zero.

Definition The function f is *piecewise continuous* on an interval (a, b) if we can divide (a, b) into a finite number of sub-intervals, on each of which f is defined and continuous, and the left- and right-hand limits at the endpoints of each sub-interval exist.

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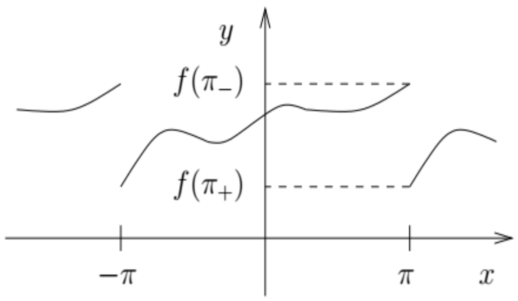
Theorem 2.2 (Convergence theorem) Let f be a periodic function with period 2π, with f and f' piecewise continuous on $(-\pi, \pi)$. Then the Fourier series of f at x converges to the value $\frac{1}{2}[f(x_+) + f(x_-)]$, *i.e.*

$$\frac{1}{2}\left[f(x_+) + f(x_-))\right] = \frac{1}{2}a_0 + \sum_{n=0}^{\infty}\left[a_n \cos(nx) + b_n \sin(nx)\right]. \tag{2.53}$$

Note that if f is continuous at x, then $f(x_+) = f(x_-) = f(x)$ so the Fourier series converges to $f(x)$.

Note that if a function is defined on an interval of length 2π, we can find the Fourier series of its periodic extension and equation (2.53) will then hold on the original interval. But we have to be careful at the end points of the interval: *e.g.* if f is defined on $(-\pi, \pi]$ then at $\pm\pi$ the Fourier Series of f converges to $\frac{1}{2}\left[f(\pi_-) + f((-\pi)_+)\right]$.



For Example 2.2 we have, by Theorem 2.2, that

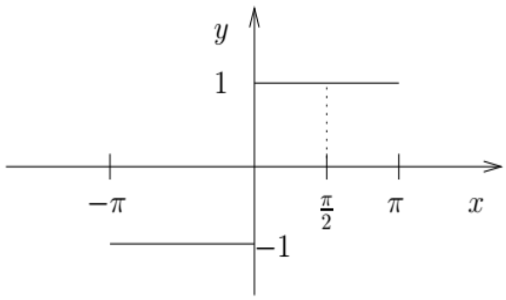
$$\frac{1}{2}\left[f(x_+) + f(x_-)\right] = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{2m+1}\sin[(2m+1)x], \tag{2.54}$$

where both sides reduce to zero at $x = 0, \pm\pi$. At $x = \pi/2$ we obtain

$$1 = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{2m+1}\sin\left[\frac{(2m+1)\pi}{2}\right] = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}, \tag{2.55}$$

and hence

$$\frac{\pi}{4} = \sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}. \tag{2.56}$$



Rate of convergence

When using Fourier series in practical situations, we often need to truncate the series at some finite value of n. In this case, we would be interested in questions such as how good is the convergence? Also, what about the speed of the convergence? In general, the more derivatives f has, the faster the convergence. We can roughly say that if the discontinuity is in the p^{th} derivative, then a_n, b_n decay like n^{-p-1}.

Lemma 2.3 Assume that F is continuous on $[-\pi, \pi]$, and $F' = f$ is piecewise continuous on $(-\pi, \pi)$. Let a_n, b_n be the Fourier coefficients of f and A_n, B_n be the Fourier coefficients of F. Then, $A_n = -b_n/n$ and $B_n = a_n/n$.

Proof. The proof is an integration by parts, and is not shown here. □

In fact, this is best seen using *complex* Fourier coefficients, $c_n := a_n + ib_n$. Then the lemma says that

$$c_n(f') = -inc_n(f). \tag{2.57}$$

This can be iterated and used to solve ODEs. For example, if f is a 2π periodic function, with Fourier coefficients $c_n(f)$, and y is the solution of the differential equation

$$y^{(5)}(x) + a_4 y^{(4)}(x) + a_3 y^{(3)}(x) + a_2 y^{(2)}(x) + a_1 y^{(1)}(x) + a_0 y(x) = f(x), \tag{2.58}$$

then the Fourier coefficients of y, $c_n(y)$, $n \geq 0$, are given by

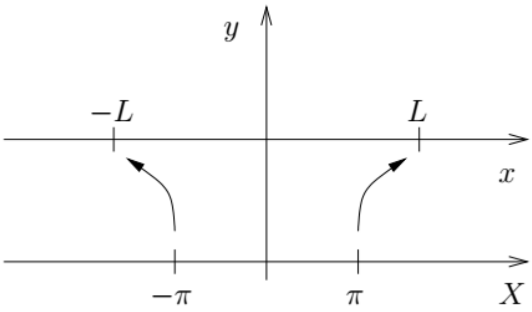
$$\left[(-in)^5 + a_4(-in)^4 + a_3(-in)^3 + a_2(-in)^2 + a_1(-in) + a_0\right] c_n(y) = c_n(f). \tag{2.59}$$

Gibbs Phenomenon

As can be seen in Figure 2.1, at a point of discontinuity, the partial sums always overshoot the limiting values. This overshoot does not tend to zero as more terms are taken, but the width of the overshooting region does tend to zero. This is known as the Gibbs Phenomenon.

Functions of any period

Consider a function f of period $2L$ $(L > 0)$. We want a series in $\cos(n\pi x/L)$ and $\sin(n\pi x/L)$. To do this we make the transformation $X = \pi x/L$.



Formally, we define $g(X) = f(x) = f(LX/\pi)$ so that

$$g(X + 2\pi) = f\left(\frac{L(X + 2\pi)}{\pi}\right) = f\left(\frac{LX}{\pi} + 2L\right) = f\left(\frac{LX}{\pi}\right) = g(X), \qquad (2.60)$$

and g is 2π-periodic. Hence the previous theory holds for g, *i.e.* if we can write

$$g(X) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(nX) + b_n \sin(nX)\right], \qquad (2.61)$$

then

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(X)\cos(nX)\,\mathrm{d}X, \\ &= \frac{1}{\pi}\int_{-L}^{L} g\left(\frac{\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\frac{\pi}{L}\,\mathrm{d}x, \\ &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x, \qquad (2.62) \end{aligned}$$

and

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(X)\sin(nX)\,\mathrm{d}X, \\ &= \frac{1}{\pi}\int_{-L}^{L} g\left(\frac{\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\frac{\pi}{L}\,\mathrm{d}x, \\ &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x. \qquad (2.63) \end{aligned}$$

So if we can write

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right], \tag{2.64}$$

then (2.61) holds, so

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x, \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x. \tag{2.65}$$

The series in equation (2.64) is called the Fourier series for f and a_n and b_n are the Fourier coefficients of f. Again, we use $\sim$ if we do not know whether or not it converges. By Theorem 2.2, under suitable conditions the series in equation (2.61) converges to

$$\frac{g(X_+) + g(X_-)}{2}, \tag{2.66}$$

so we obtain

Theorem 2.4 Let f be a periodic function of period $2L$ which is sufficiently well-behaved. Then the Fourier series of f at x converges to

$$\frac{f(x_+) + f(x_-)}{2}, \tag{2.67}$$

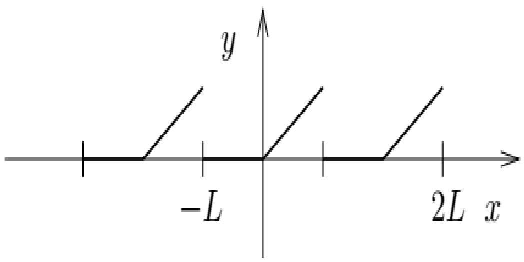
so equation (2.64) holds at any point where f is continuous.

Example 2.3 Find the Fourier series of the $2L$-periodic extension of

$$f(x) = \begin{cases} x & x \in (0, L], \\ 0 & x \in (-L, 0]. \end{cases} \tag{2.68}$$

Hence show that

$$\frac{\pi^2}{8} = \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}. \tag{2.69}$$



We have

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x = \frac{1}{L}\int_{0}^{L} x\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x = \frac{L[(-1)^n - 1]}{n^2\pi^2}, \quad n \neq 0, \tag{2.70}$$

as in Example 2.1. So we have $a_{2m} = 0$ for $m > 0$ and

$$a_{2m+1} = \frac{-2L}{(2m+1)^2\pi^2}. \tag{2.71}$$

For a_0 we calculate

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,\mathrm{d}x = \frac{1}{L}\int_{0}^{L} x\,\mathrm{d}x = \frac{L}{2}, \tag{2.72}$$

and for b_n

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x, \\ &= \frac{1}{L}\int_{0}^{L} x\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x, \\ &= \frac{1}{L}\left(\left[-\frac{Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]_0^L + \int_0^L \frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x\right), \\ &= \frac{1}{L}\left(-\frac{L^2(-1)^n}{n\pi} + \left[\frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)\right]_0^L\right), \\ &= (-1)^{n+1}\frac{L}{n\pi}. \end{aligned}$$

So

$$f(x) \sim \frac{L}{4} + \sum_{m=0}^{\infty}\frac{-2L}{(2m+1)^2\pi^2}\cos\left[\frac{(2m+1)\pi x}{L}\right] + \sum_{m=1}^{\infty}(-1)^{m+1}\frac{L}{m\pi}\sin\left(\frac{m\pi x}{L}\right). \tag{2.73}$$

By Theorem 2.4, if $x \in [0, L)$ we obtain

$$x = \frac{L}{4} + \sum_{m=0}^{\infty}\frac{-2L}{(2m+1)^2\pi^2}\cos\left[\frac{(2m+1)\pi x}{L}\right] + \sum_{m=1}^{\infty}(-1)^{m+1}\frac{L}{m\pi}\sin\left(\frac{m\pi x}{L}\right). \tag{2.74}$$

because f is continuous on $[0, L)$. If we put $x = 0$ we calculate

$$0 = \frac{L}{4} + \sum_{m=0}^{\infty} \frac{-2L}{(2m+1)^2\pi^2}, \tag{2.75}$$

which proves (2.69). If we set $x = L$ in equation (2.73) we obtain

$$\frac{f(L_+) + f(L_-)}{2} = \frac{L}{4} + \sum_{m=0}^{\infty} \frac{-2L}{(2m+1)^2\pi^2} \cos[(2m+1)\pi], \tag{2.76}$$

giving

$$\frac{0 + L}{2} = \frac{L}{4} + \sum_{m=0}^{\infty} \frac{-2L}{(2m+1)^2\pi^2}, \tag{2.77}$$

which gives equation (2.69) again.

2.4.1 Sine and cosine series

Given a function f defined on $[0, L]$ we require an expansion with only cosine terms or only sine terms. This will be done by extending f to be even (for only cosine terms) or odd (for only sine terms) on $(-L, L]$ and then extending to a $2L$-period function. The series obtained will then be valid on $(0, L)$.

Definition If f is defined on $[0, L]$, the *even extension* for f, denoted by f_e, is the periodic extension of

$$f_e(x) = \begin{cases} f(x) & x \in [0, L], \\ f(-x) & x \in (-L, 0), \end{cases} \tag{2.78}$$

so that we have $f_e(x) = f_e(-x)$ for all x. Thus:

$$f_e(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \tag{2.79}$$

where

$$a_n = \frac{1}{L} \int_{-L}^{L} f_e(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x, \tag{2.80}$$

is called the *Fourier cosine series* of f.

Definition The *odd extension* for f, denoted by f_o, is the periodic extension of

$$f_o(x) = \begin{cases} f(x) & x \in [0, L], \\ -f(-x) & x \in (-L, 0), \end{cases} \tag{2.81}$$

so that $f_o(x) = -f_o(-x)$ for all $x \neq nL$. Similarly,

$$f_o(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \tag{2.82}$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x,$$

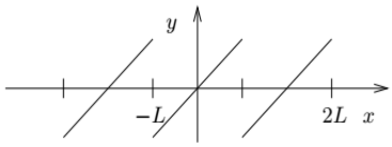
is called the *Fourier sine series* for f.

Note. For f_o to really be odd we must have $f_o(0) = 0$ and also $f_o(L) = -f_o(-L) = -f_o(L)$ (the last equality follows from periodicity) giving $f_o(L) = 0$ and therefore $f_o(nL) = 0$ for all $n \in \mathbb{Z}$. However, the value at of f at these isolated points does not affect the Fourier series.

Example 2.4 Find the Fourier sine and cosine expansions of $f(x) = x$ for $x \in [0, L]$.

Sine expansion The odd extension is defined by

$$f_o(x) = \begin{cases} x & x \in [0, L], \\ -(-x) & x \in (-L, 0). \end{cases} \tag{2.83}$$



In this case

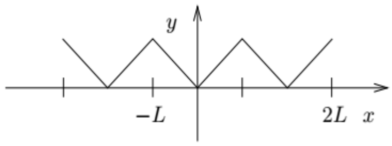
$$b_n = \frac{2}{L}\int_0^L x \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x = (-1)^{n+1}\frac{2L}{n\pi}, \tag{2.84}$$

as in Example 2.3. For $x \in [0, L)$ we therefore obtain

$$x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2L}{n\pi}\sin\left(\frac{n\pi x}{L}\right). \tag{2.85}$$

Cosine expansion The even extension is given by

$$f_e(x) = \begin{cases} x & x \in [0, L], \\ -x & x \in [-L, 0). \end{cases} \tag{2.86}$$



Now,

$$a_0 = \frac{2}{L}\int_0^L x \,\mathrm{d}x = L, \tag{2.87}$$

and

$$a_n = \frac{2}{L}\int_0^L x \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \begin{cases} 0 & n = 2m \text{ even}, \\ \frac{-4L}{(2m+1)^2\pi^2} & n = 2m+1 \text{ odd}. \end{cases} \tag{2.88}$$

Thus for $x \in [0, L]$ we get

$$x = \frac{L}{2} + \sum_{m=0}^{\infty} -\frac{4L}{(2m+1)^2\pi^2}\cos\left[\frac{(2m+1)\pi x}{L}\right]. \tag{2.89}$$

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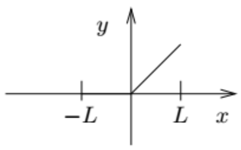
Recall that (2.74) is the Fourier series of

$$h(x) = \begin{cases} x & x \in [0, L], \\ 0 & x \in (-L, 0). \end{cases} \tag{2.90}$$



Looking at the results from the previous example this indicates that the Fourier series of $f(x) + g(x)$ equals the Fourier series of $f(x)$ plus the Fourier series of $g(x)$.