

# FOURIER SERIES AND PDEs

## LECTURE 3

### FOURIER SERIES FOR FUNCTIONS OF PERIOD $2\pi$

Let  $f$  be a function of period  $2\pi$ . We would like to get an expansion for  $f$  of the form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)], \quad (2.19)$$

where the  $a_n$  and  $b_n$  are constants. Remember that  $\sin(nx)$  and  $\cos(nx)$  are periodic with period  $2\pi$ . We have two questions to answer.

**Question 1:** if equation (2.19) is true, can we find the  $a_n$  and  $b_n$  in terms of  $f$ ?

**Question 2:** with these  $a_n, b_n$ , when, if ever, is equation (2.19) true?

### Question 1

Suppose equation (2.19) is true and that we can integrate it term by term, *i.e.*

$$\int_{-\pi}^{\pi} f(x) dx = \frac{1}{2}a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left[ a_n \int_{-\pi}^{\pi} \cos(nx) dx + b_n \int_{-\pi}^{\pi} \sin(nx) dx \right]. \quad (2.20)$$

Since

$$\int_{-\pi}^{\pi} dx = 2\pi, \quad \int_{-\pi}^{\pi} \cos(nx) dx = 0, \quad \int_{-\pi}^{\pi} \sin(nx) dx = 0, \quad (2.21)$$

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we must have

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx. \quad (2.22)$$

Thus if equation (2.19) is true, then we know  $a_0$ .

Note that due to the properties of periodic functions, we could use  $\int_0^{2\pi} f(x) dx$  instead of  $\int_{-\pi}^{\pi} f(x) dx$  in the preceding. More generally, we could integrate over any interval of length  $2\pi$ .

**Lemma 2.1** Let  $n, m \in \mathbb{N} \setminus 0$ . We then have the following equalities:

$$\int_{-\pi}^{\pi} \sin(mx) \cos(nx) dx = 0, \quad (2.23)$$

$$\int_{-\pi}^{\pi} \sin(mx) \sin(nx) dx = \pi \delta_{nm}, \quad (2.24)$$

$$\int_{-\pi}^{\pi} \cos(mx) \cos(nx) dx = \pi \delta_{nm}, \quad (2.25)$$

where  $\delta_{nm}$  is the Kronecker delta defined by

$$\delta_{nm} = \begin{cases} 0 & n \neq m, \\ 1 & n = m. \end{cases} \quad (2.26)$$

*Proof.* Equation (2.23) is trivial as  $\sin(mx) \cos(nx)$  is odd. For equation (2.24) we compute, for  $n \neq m$ ,

$$\begin{aligned} \int_{-\pi}^{\pi} \sin(mx) \sin(nx) dx &= \frac{1}{2} \int_{-\pi}^{\pi} [-\cos\{(m+n)x\} + \cos\{(m-n)x\}] dx, \\ &= \frac{1}{2} \left[ \frac{-\sin\{(m+n)x\}}{m+n} + \frac{\sin\{(m-n)x\}}{m-n} \right]_{-\pi}^{\pi}, \\ &= 0. \end{aligned} \quad (2.27)$$

If  $n = m$  we have

$$\int_{-\pi}^{\pi} \sin(nx) \sin(mx) dx = \int_{-\pi}^{\pi} \left[ \frac{1 - \cos(2nx)}{2} \right] dx = \frac{1}{2} \left[ x - \frac{\sin(2nx)}{2n} \right]_{-\pi}^{\pi} = \pi. \quad (2.28)$$

Similar computations yield equation (2.25). □

Thus, to find  $a_n$  and  $b_n$ , we assume equation (2.19) is true. Multiplying both sides by  $\cos(mx)$  and integrating term-wise gives

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \cos(mx) dx &= \frac{1}{2} a_0 \int_{-\pi}^{\pi} \cos(mx) dx \\ &\quad + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos(nx) \cos(mx) dx \\ &\quad + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin(nx) \cos(mx) dx. \end{aligned} \quad (2.29)$$

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The first term on the right-hand side is trivially zero for  $m \neq 0$ . Using Lemma 2.1 for the remaining terms gives

$$\int_{-\pi}^{\pi} f(x) \cos(mx) \, dx = \sum_{n=1}^{\infty} a_n \pi \delta_{nm} = \pi a_m, \quad (2.30)$$

and hence

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(mx) \, dx. \quad (2.31)$$

Note that this also holds for  $m = 0$  (which is the reason for the factor of  $1/2$ ).

Multiplying equation (2.19) by  $\sin(mx)$  and integrating term-wise, we similarly obtain

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx) \, dx. \quad (2.32)$$

**Definition** Suppose  $f$  is such that

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx, \quad (2.33)$$

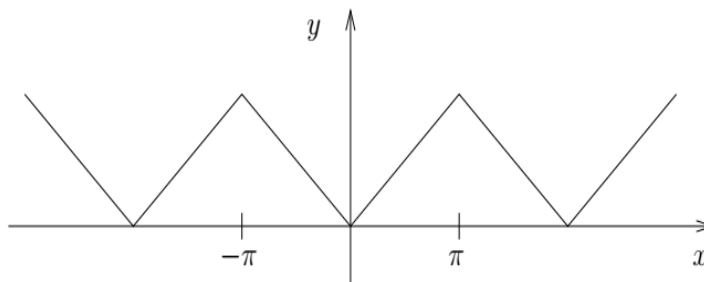
exist. Then we shall write

$$f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)], \quad (2.34)$$

and call the series on the right-hand side the *Fourier series* for  $f$ , whether or not it converges to  $f$ . The constants  $a_n$  and  $b_n$  are called the *Fourier coefficients* of  $f$ .

**Example 2.1** Find the Fourier series of the function  $f$  which is periodic with period  $2\pi$  and such that

$$f(x) = |x|, \quad x \in (-\pi, \pi]. \quad (2.35)$$



To find  $a_n, b_n$  first notice that  $f$  is even, so  $f(x) \sin(nx)$  is odd and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx = 0, \quad (2.36)$$

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for every  $n$ . Also,  $f(x) \cos(nx)$  is even, so

$$\begin{aligned}
 a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx, \\
 &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) \, dx, \\
 &= \frac{2}{\pi} \int_0^{\pi} x \cos(nx) \, dx, \\
 &= \frac{2}{\pi} \left( \left[ \frac{x \sin(nx)}{n} \right]_0^{\pi} - \int_0^{\pi} \frac{\sin(nx)}{n} \, dx \right), \\
 &= -\frac{2}{\pi} \left( \left[ \frac{-\cos(nx)}{n^2} \right]_0^{\pi} \right), \\
 &= \frac{2}{\pi} \left( \frac{\cos(n\pi) - \cos(0)}{n^2} \right), \\
 &= \frac{2 [(-1)^n - 1]}{\pi n^2}. \tag{2.37}
 \end{aligned}$$

Note that this is not valid for  $n = 0$ . In fact,  $a_0 = \pi$ . If  $n$  is even,  $n = 2m$  say, we have

$$a_{2m} = \frac{2((-1)^{2m} - 1)}{\pi(2m)^2} = 0. \tag{2.38}$$

If  $n$  is odd,  $n = 2m + 1$  say, we obtain

$$a_{2m+1} = \frac{2(-1 - 1)}{\pi(2m + 1)^2} = \frac{-4}{\pi(2m + 1)^2}. \tag{2.39}$$

### 2.2.2 Sine and cosine series

Let  $f$  be  $2\pi$ -periodic. If  $f$  is odd then

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx), \tag{2.40}$$

where

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(s) \sin(ns) \, ds = \frac{2}{\pi} \int_0^{\pi} f(s) \sin(ns) \, ds, \tag{2.41}$$

*i.e.*  $f$  has a Fourier sine series. In this case  $a_n = 0$  because  $f(x) \cos(nx)$  is odd. This is also true if  $f(x) = -f(-x)$  for  $x \neq n\pi$ ,  $n \in \mathbb{Z}$ , *i.e.*  $f$  is odd apart from the end points and zero.

If  $f$  is even then

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx), \tag{2.42}$$

where

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(s) \cos(ns) \, ds, \tag{2.43}$$

*i.e.*  $f$  has a Fourier cosine series.

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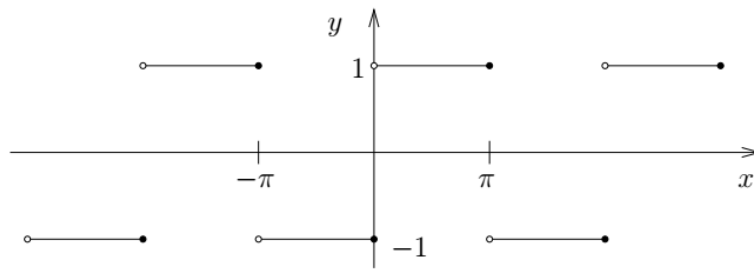
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### Question 2

Recall Question 2: with these  $a_n, b_n$ , when, if ever, is equation (2.19) true? Consider what happens in the following example.

**Example 2.2** Consider the Fourier series of the function  $f$  which is periodic with period  $2\pi$  and such that

$$f(x) = \begin{cases} 1 & 0 < x \leq \pi, \\ -1 & -\pi < x \leq 0. \end{cases} \quad (2.44)$$



Note that  $f$  is odd, so we can conclude that  $f(x) \cos(nx)$  is odd, giving  $a_n = 0$  without computation. On the other hand,  $f(x) \sin(nx)$  is even, so

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx = \frac{2}{\pi} \int_0^{\pi} \sin(nx) dx = -\frac{2[(-1)^n - 1]}{n\pi}, \quad (2.45)$$

*i.e.*

$$b_{2m} = 0, \quad b_{2m+1} = \frac{4}{(2m+1)\pi}, \quad (2.46)$$

and hence

$$f(x) \sim \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{1}{2m+1} \sin[(2m+1)x]. \quad (2.47)$$

Consider Question 2 for this case: when is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]? \quad (2.48)$$

Recall that

$$\frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\sin[(2m+1)x]}{2m+1}, \quad (2.49)$$

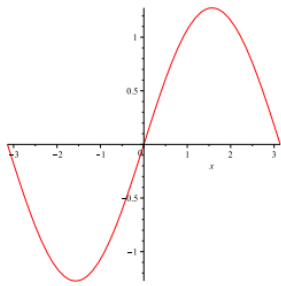
means  $\lim_{n \rightarrow \infty} s_n(x)$  where

$$s_n(x) = \frac{4}{\pi} \sum_{m=0}^n \frac{\sin[(2m+1)x]}{2m+1}. \quad (2.50)$$

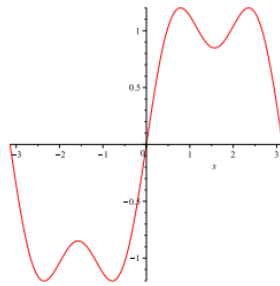
The question is therefore, does  $s_n(x)$  converge for each  $x$ ? If it does, is the limit  $f(x)$ ? Some partial sums,  $s_n$ , are plotted in Figure 2.1.

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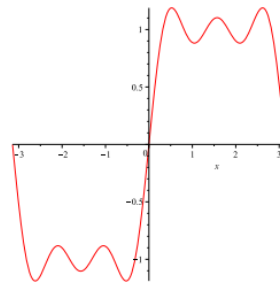
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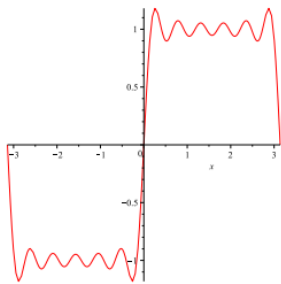
(a)  $s_0$



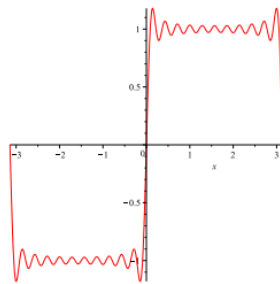
(b)  $s_1$



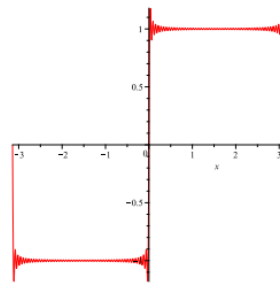
(c)  $s_2$



(d)  $s_5$



(e)  $s_{10}$



(f)  $s_{50}$