FOURIER SERIES AND PDEs
LECTURE 2

In the following chapters, we will look at methods for solving the PDEs described in Chapter 1. In order to incorporate general initial or boundary conditions into our solutions, it will be necessary to have some understanding of Fourier series.

For example, we can see that the series

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right], \tag{2.1}$$

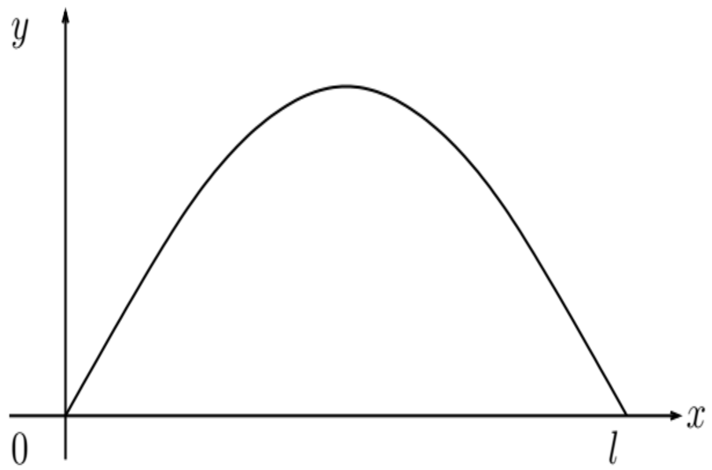
is a solution of the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad x \in [0, L], \quad t \geq 0, \tag{2.2}$$

which satisfies the boundary conditions

$$y(0,t) = 0 = y(L,t). \tag{2.3}$$

We may view $y(x,t)$ as the solution of the problem which models a vibrating string of length L pinned at both ends, *e.g.* a guitar string.



We would like to find a solution with initial conditions

$$y(x,0) = \alpha \sin\left(\frac{\pi x}{L}\right), \qquad \frac{\partial y}{\partial t}(x,0) = 0, \tag{2.4}$$

and we do this by calculating A_n and B_n as follows: from equation (2.1) we have

$$y(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right), \tag{2.5}$$

and

$$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} B_n \left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi x}{L}\right). \tag{2.6}$$

Hence, we want A_n, B_n such that

$$\alpha \sin\left(\frac{\pi x}{L}\right) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right), \qquad 0 = \sum_{n=1}^{\infty} B_n \left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi x}{L}\right). \tag{2.7}$$

By inspection we see that $A_1 = \alpha$, $A_n = 0$ for $n \neq 1$ and $B_n = 0\ \forall n$. Thus, for these initial conditions, the solution is

$$y(x,t) = \alpha \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{\pi c t}{L}\right). \tag{2.8}$$

If we would like to take more general initial conditions

$$y(x,0) = f(x), \qquad \frac{\partial y}{\partial t}(x,0) = g(x), \tag{2.9}$$

we need to find $\{A_n, B_n\}$ such that

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right), \qquad g(x) = \sum_{n=1}^{\infty} B_n \left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi x}{L}\right). \tag{2.10}$$

These are known the Fourier sine series of the functions f and g.

Periodic, even and odd functions

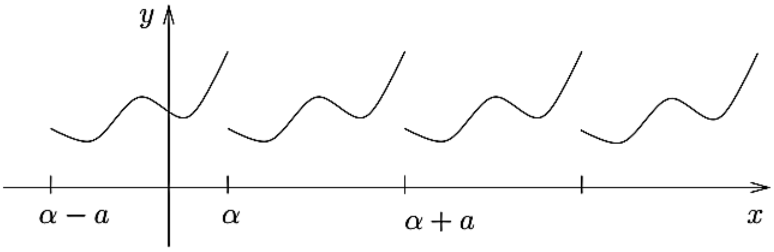
Definition f is a *periodic function* if there is an $a > 0$ such that

$$f(x+a) = f(x), \qquad \forall x \in \mathbb{R}. \tag{2.11}$$

If this is the case a is called a *period* for f. Note that the period is not unique, but if there is a smallest such a, it is called the *prime period* of f.

Notes.

1. Observe that this means that $f(x) = c$ for c constant does not have a prime period.
2. Examples of periodic functions are $\sin x$ with prime period 2π and $\cos(2\pi x/a)$ with prime period a. Examples of non-periodic functions are x and x^2.
3. Observe that if f is defined on the half-open interval $(\alpha, \alpha + a]$ we can extend it to be a periodic function by demanding it is periodic with period a. This is called a periodic extension.



Definition Formally, we define the *periodic extension*, F, of f as follows: given $x \in \mathbb{R}$ there exists a unique integer m such that $x - ma \in (\alpha, \alpha + a]$. If we then set $F(x) = f(x - ma)$, we can see that F is periodic with period a.

Properties of periodic functions

If f, g are periodic functions with period a, then:

1. f, g are also periodic functions with period na for any $n \in \mathbb{N}$;
2. for any $\alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ is periodic with period a;
3. fg is periodic with period a;
4. for any $\lambda > 0$, $f(\lambda x)$ is periodic with period a/λ,

$$f(\lambda(x + a/\lambda)) = f(\lambda x + a) = f(\lambda x); \tag{2.12}$$

5. for any $\alpha \in \mathbb{R}$,

$$\int_0^a f(x)\,\mathrm{d}x = \int_\alpha^{\alpha+a} f(x)\,\mathrm{d}x, \tag{2.13}$$

since

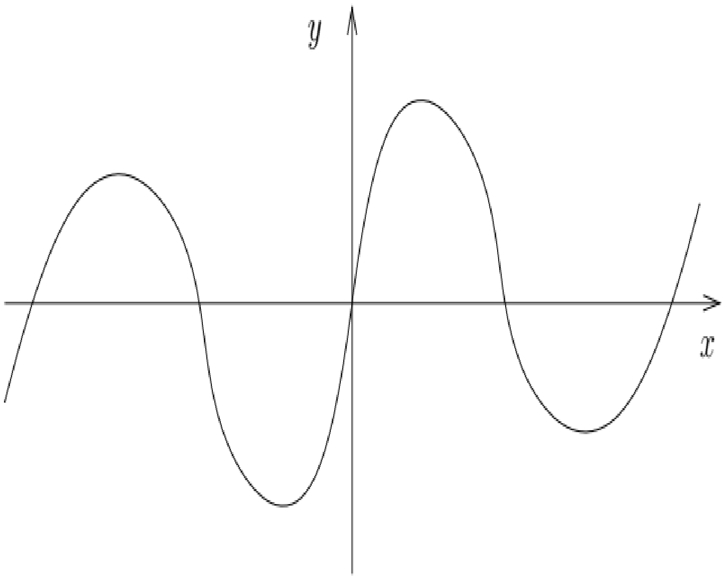
$$\int_\alpha^{\alpha+a} f(x)\,\mathrm{d}x = \int_\alpha^a f(x)\,\mathrm{d}x + \int_a^{\alpha+a} f(x)\,\mathrm{d}x. \tag{2.14}$$

Odd and even functions

Definition A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *odd* if

$$f(x) = -f(-x), \qquad \forall x \in \mathbb{R}. \tag{2.15}$$

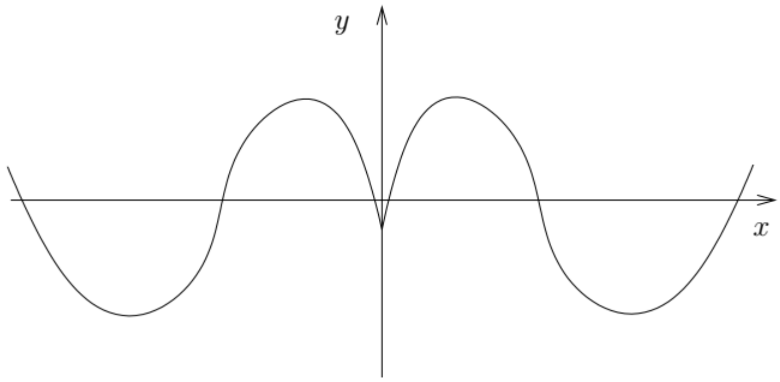
For example, $\sin(\lambda x)$ for $\lambda \in \mathbb{R}$ and x^{2n+1} for $n \in \mathbb{N}$ are both odd functions.



Definition A function $g : \mathbb{R} \to \mathbb{R}$ is said to be *even* if

$$g(x) = g(-x), \qquad \forall x \in \mathbb{R}. \tag{2.16}$$

Examples of even functions are $\cos(\lambda x)$ for $\lambda \in \mathbb{R}$ and x^{2n} for $n \in \mathbb{N}$.



Notes. If f, f_1 are odd functions and g, g_1 are even functions then:

1. $f(0) = 0$ because $f(0) = -f(-0) = -f(0)$;
2. for any $\alpha \in \mathbb{R}$,
$$\int_{-\alpha}^{\alpha} f(x)\,\mathrm{d}x = 0; \tag{2.17}$$
3. for any $\alpha \in \mathbb{R}$,
$$\int_{-\alpha}^{\alpha} g(x)\,\mathrm{d}x = 2\int_{0}^{\alpha} g(x)\,\mathrm{d}x; \tag{2.18}$$
4. the functions $h(x) = f(x)g(x)$, $h_1(x) = f(x)f_1(x)$ and $h_2(x) = g(x)g_1(x)$ are odd, even and even, respectively.