FOURIER SERIES AND PDEs
LECTURE 1

INITIAL AND BOUNDARY VALUE PROBLEM

Consider a second-order ordinary differential equation (ODE)

$$y'' = f(x, y, y'), \tag{1.1}$$

where $y' = \mathrm{d}y/\mathrm{d}x$ and $y'' = \mathrm{d}^2y/\mathrm{d}x^2$. The problem is to find $y(x)$, subject to appropriate additional information.

Initial-value problem (IVP)

Suppose that $y(a) = p$ and $y'(a) = q$ are prescribed.

$$y = p + q(x - a).$$

Boundary-value problem (BVP)

Suppose that $y(x)$ is defined on an interval $[a, b]$ and $y(a) = A$ and $y(b) = B$ are prescribed.

Existence and uniqueness

Recall that solutions may not exist, or if they exist they may not be unique.

IVP: $y'' = 6y^{\frac{1}{3}}$, $y(0) = 0$, $y'(0) = 0$ has solutions $y(x) = 0$, $y(x) = x^3$ (non-uniqueness);

BVP1: $y'' + y = 0$, $y(0) = 1$, $y(2\pi) = 0$ has no solution (non-existence);

BVP2: $y'' + y = 0$, $y(0) = 0$, $y(2\pi) = 0$ has infinitely many solutions, $y(x) = c\sin x$, where c is an arbitrary constant (non-uniqueness).

Some preliminaries

We state, but do not prove, two preliminary results.

Theorem 1.1 (Leibniz's Integral Rule) Let $F(x,t)$ and $\partial F/\partial t$ be continuous in both x and t in some region of the (x,t) plane including $(t,x) \in [t_0,t_1] \times [a(t),b(t)]$, and the functions $a(t)$ and $b(t)$ and their derivatives be continuous for $t \in [t_0,t_1]$. Then

$$G(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} F(x,t)\,\mathrm{d}x = b'(t)F(b(t),t) - a'(t)F(a(t),t) + \int_{a(t)}^{b(t)} \frac{\partial F(x,t)}{\partial t}\,\mathrm{d}x. \quad (1.2)$$

As a result, if $a(t)$ and $b(t)$ are constants with

$$G(t) = \int_a^b F(x,t)\,\mathrm{d}x, \quad (1.3)$$

then

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \int_a^b \frac{\partial F(x,t)}{\partial t}\,\mathrm{d}x. \quad (1.4)$$

Lemma 1.2 If $f(x)$ is continuous then

$$\frac{1}{h}\int_a^{a+h} f(x)\,\mathrm{d}x \to f(a) \text{ as } h \to 0.$$

Note that

$$\frac{G(t+h)-G(t)}{h} = \int_a^b \frac{F(x,t+h)-F(x,t)}{h}\mathrm{d}x, \quad (1.5)$$

and the integrand tends to $\partial F(x,t)/\partial t$ as $h \to 0$.

The equations we shall study

It is proposed to study three linear second-order partial differential equations (PDEs) that have applications throughout the physical sciences.

The heat equation

Also known as the diffusion equation, we will find $T(x,t)$ such that

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}, \tag{1.6}$$

where, for example, $T(x,t)$ is a temperature at position x and time t, and κ is a positive constant—the *thermal diffusivity*.

The wave equation

Here, we will look at finding $y(x,t)$ such that

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \tag{1.7}$$

where, for example, $y(x,t)$ is the transverse displacement of a stretched string at position x and time t, and c is a positive constant—the *wave speed*.

Laplace's equation

In this case the problem is to find $T(x,y)$ such that

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \tag{1.8}$$

where, for example, $T(x,y)$ may be a temperature and x and y are Cartesian coordinates in the plane. In this case, Laplace's equation models a two-dimensional system at steady state in time: in three space-dimensions the temperature $T(x,y,z,t)$ satisfies the heat equation

$$\frac{\partial T}{\partial t} = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right). \tag{1.9}$$

Note that equation (1.9) reduces to (3.8) if T is independent of y and z. If the temperature field is *static*, T is independent of time, t, and is a solution of *Laplace's equation in* $\mathbb{R}^3$,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0, \tag{1.10}$$

and, in the special case in which T is also independent of z, of *Laplace's equation in* $\mathbb{R}^2$,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0. \tag{1.11}$$