WEEK 3: Yield Curve Theories and Measuring Price Sensitivity to Changes in Yield

TOPICS:

- **The Treasury Yield Curve**
- **Theories of the Term Structure of Interest Rates**
- **Measuring price sensitivity of bonds to change in interest rates**
- **Effective duration**
- **Macaulay duration and Modified duration**
- **Convexity and convexity adjustment**

The Treasury Yield Curve

Given that US Treasury securities do not expose investors to credit risk, market participants look at the yield offered on an on-the-run Treasury security as the minimum interest rate required on a non-Treasury security with the same maturity.

The longer the maturity the higher the yield and is referred to as an upward sloping yield curve. Since this is the most typical shape for the Treasury yield curve, it is also referred to as a normal yield curve. Other relationships have been observed. An inverted yield curve indicates that the longer the maturity, the lower the yield. For a flat yield curve the yield is approximately the same regardless of maturity.

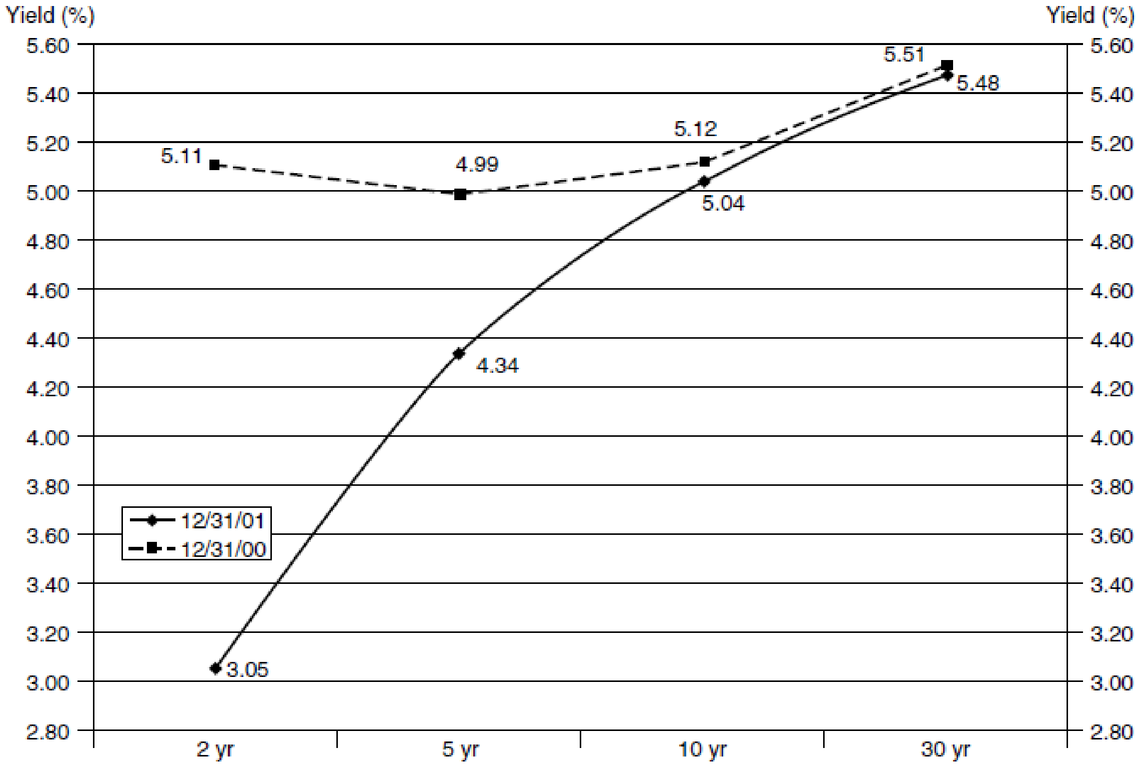
Exhibit 2 provides a graphic example of the variants of these shapes and also shows how a yield curve can change over time. In the exhibit, the yield curve at the beginning of 2001 was inverted up to the 5-year maturity but was upward sloping beyond the 5-year maturity.

By December 2001, all interest rates had declined. As seen in the exhibit, interest rates less than the 10-year maturity dropped substantially more than longer-term rates resulting in an upward sloping yield curve.

The number of on-the-run securities available in constructing the yield curve has decreased over the last two decades. While the 1-year and 30-year yields are shown in the February 8, 2002 yield curve, as of this writing there is no 1-year Treasury bill and the maturity of the 30-year Treasury bond (the last one issued before suspension of the issuance of 30-year Treasury bonds) will decline over time. To get a yield for maturities where no on-the-run Treasury issue exists, it is necessary to interpolate from the yield of two on-the-run issues.

Several methodologies are used in practice. (The simplest is just a linear interpolation.) Thus, when market participants talk about a yield on the Treasury yield curve that is not one of the available on-the-run maturities—for example, the 8-year yield—it is only an approximation.

EXHIBIT 2 U.S. Treasury Yield Curve: December 2000 and December 2001



Source: Lehman Brothers Fixed Income Research, *Global Fixed Income Strategy "Playbook,"* January 2002.

Theories of the Term Structure of Interest Rates

What information does the yield curve reveal? How can we explain and interpret changes in the yield curve? These questions are of great interest to anyone concerned with such tasks as the valuation of multi-period securities, economic forecasting, and risk management. Theories of the term structure of interest rates address these questions. Here we introduce the three main theories or explanations of the term structure. We shall present these theories intuitively. The three main term structure theories are:

- the pure expectations theory (unbiased expectations theory)
- the liquidity preference theory (or liquidity premium theory)
- the market segmentation theory

Pure Expectations Theory

The pure expectations theory makes the simplest and most direct link between the yield curve and investors' expectations about future interest rates, and, because long-term interest rates are plausibly linked to investor expectations about future inflation, it also opens the door to some interesting economic interpretations.

The pure expectations theory explains the term structure in terms of expected future short-term interest rates. According to the pure expectations theory, the market sets the yield on a two-year bond so that the return on the two-year bond is approximately equal to the return on a one-year bond plus the expected return on a one-year bond purchased one year from today.

Under this theory, a rising term structure indicates that the market expects short-term rates to rise in the future. For example, if the yield on the two-year bond is higher than the yield on the one-year bond, according to this theory, investors expect the one-year rate a year from now to be sufficiently higher than the one-year rate available now so that the two ways of investing for two years have the same expected return. Similarly, a flat term structure reflects an expectation that future short-term rates will be unchanged from today's short-term rates, while a falling term structure reflects an expectation that future short-term rates will decline.

This is summarized below:

Shape of term structure	Implication according to pure expectations theory
upward sloping (normal)	rates expected to rise
downward sloping (inverted)	rates expected to decline
flat	rates not expected to change

The implications above are the broadest interpretation of the theory. How does the pure expectations theory explain a humped yield curve? According to the theory, this can result when investors expect the returns on one-year securities to rise for a number of years, then fall for a number of years.

The relationships that the table above illustrates suggest that the shape of the yield curve contains information regarding investors' expectations about future inflation. A pioneer of the theory of interest rates (the economist Irving Fisher) asserted that interest rates reflect the sum of a relatively stable real rate of interest plus a premium for expected inflation. Under this hypothesis, if short-term rates are expected to rise, investors expect inflation to rise as well.

An upward (downward) sloping term structure would mean that investors expected rising (declining) future inflation. Much economic discussion in the financial press and elsewhere is based on this interpretation of the yield curve.

The shortcoming of the pure expectations theory is that it assumes investors are indifferent to interest rate risk and any other risk factors associated with investing in bonds with different maturities.

Liquidity Preference Theory

The liquidity preference theory asserts that market participants want to be compensated for the interest rate risk associated with holding longer term bonds. The longer the maturity, the greater the price volatility when interest rates change and investors want to be compensated for this risk. According to the liquidity preference theory, the term structure of interest rates is determined by (1) expectations about future interest rates and (2) a yield premium for interest rate risk.6 Because interest rate risk increases with maturity, the liquidity preference theory asserts that the yield premium increases with maturity.

Consequently, based on this theory, an upward-sloping yield curve may reflect expectations that future interest rates either (1) will rise, or (2) will be unchanged or even fall, but with a yield premium increasing with maturity fast enough to produce an upward sloping yield curve.

Thus, for an upward sloping yield curve (the most frequently observed type), the liquidity preference theory by itself has nothing to say about expected future short-term interest rates.

For flat or downward sloping yield curves, the liquidity preference theory is consistent with a forecast of declining future short-term interest rates, given the theory's prediction that the yield premium for interest rate risk increases with maturity.

Because the liquidity preference theory argues that the term structure is determined by both expectations regarding future interest rates and a yield premium for interest rate risk, it is referred to as biased expectations theory.

Market Segmentation Theory

Proponents of the market segmentation theory argue that within the different maturity sectors of the yield curve the supply and demand for funds determine the interest rate for that sector. That is, each maturity sector is an independent or segmented market for purposes of determining the interest rate in that maturity sector.

Thus, positive sloping, inverted, and humped yield curves are all possible. In fact, the market segmentation theory can be used to explain any shape that one might observe for the yield curve. Let's understand why proponents of this theory view each maturity sector as independent or segmented. In the bond market, investors can be divided into two groups based on their return needs: investors that manage funds versus a broad-based bond market index and those that manage funds versus their liabilities. The easiest case is for those that manage funds against liabilities. Investors managing funds where liabilities represent the benchmark will restrict their activities to the maturity sector that provides the best match with the maturity of their liabilities.7 This is the basic principle of asset-liability management. If these investors invest funds outside of the maturity sector that provides the best match against liabilities, they are exposing themselves to the risks associated with an asset-liability mismatch. For example, consider the manager of a defined benefit pension fund. Since the liabilities of a defined benefit pension fund are long-term, the manager will invest in the long-term maturity sector of the bond market. Similarly, commercial banks whose liabilities are typically short-term focus on short-term fixed-income investments. Even if the rate on long-term bonds were considerably more attractive than that on short-term investments, according to the market segmentation theory commercial banks will restrict their activities to investments at the short end of the yield curve. Reinforcing this notion of a segmented market are restrictions imposed on financial institutions that prevent them from mismatching the maturity of assets and liabilities.

A variant of the market segmentation theory is the preferred habitat theory. This theory argues that investors prefer to invest in particular maturity sectors as dedicated by the nature of their liabilities. However, proponents of this theory do not assert that investors would be unwilling to shift out of their preferred maturity sector; instead, it is argued that if investors are given an inducement to do so in the form of a yield premium they will shift out of their preferred habitat. The implication of the preferred habitat theory for the shape of the yield curve is that any shape is possible.

Treasury Strips

Although the U.S. Department of the Treasury does not issue zero-coupon Treasury securities with maturity greater than one year, government dealers can synthetically create zero-coupon securities, which are effectively guaranteed by the full faith and credit of the U.S. government, with longer maturities. They create these securities by separating the coupon payments and the principal payment of a coupon-bearing Treasury security and selling them off separately.

The process, referred to as stripping a Treasury security, results in securities called Treasury strips. The Treasury strips created from coupon payments are called Treasury coupon strips and those created from the principal payment are called Treasury principal strips.

Because zero-coupon instruments have no reinvestment risk, Treasury strips for different maturities provide a superior relationship between yield and maturity than do securities on the on-the-run Treasury yield curve. The lack of reinvestment risk eliminates the bias resulting from the difference in reinvestment risk for the securities being compared. Another advantage is that the duration of a zero-coupon security is approximately equal to its maturity.

Consequently, when comparing bond issues against Treasury strips, we can compare them on the basis of duration.

The yield on a zero-coupon security has a special name: the **spot rate**. In the case of a Treasury security, the yield is called a Treasury spot rate. The relationship between maturity and Treasury spot rates is called the term structure of interest rates. Sometimes discussions of the term structure of interest rates in the Treasury market get confusing. The Treasury yield curve and the Treasury term structure of interest rates are often used interchangeably.

While there is a technical difference between the two, the context in which these terms are used should be understood.

Measuring price sensitivity of bonds to change in interest rates (full valuation, duration, convexity)

We know that the value of a bond moves in the opposite direction to a change in interest rates. If interest rates increase, the price of a bond will decrease. For a short bond position, a loss is generated if interest rates fall. However, a manager wants to know more than simply when a position generates a loss. To control interest rate risk, a manager must be able to quantify that result. What is the key to measuring the interest rate risk? It is the accuracy in estimating the value of the position after an adverse interest rate change. A valuation model determines the value of a position after an adverse interest rate move. Consequently, if a reliable valuation model is not used, there is no way to properly measure interest rate risk exposure.

There are two approaches to measuring interest rate risk—the full valuation approach and the duration/convexity approach.

THE FULL VALUATION APPROACH

The most obvious way to measure the interest rate risk exposure of a bond position or a portfolio is to re-value it when interest rates change. The analysis is performed for different scenarios with respect to interest rate changes. For example, a manager may want to measure the interest rate exposure to a 50 basis point, 100 basis point, and 200 basis point instantaneous change in interest rates. This approach requires the re-valuation of a bond or bond portfolio for a given interest rate change scenario and is referred to as the full valuation approach.

It is sometimes referred to as scenario analysis because it involves assessing the exposure to interest rate change scenarios.

To illustrate this approach, suppose that a manager has a $10 million par value position in a 9% coupon 20-year bond. The bond is option-free. The current price is 134.6722 for a yield (i.e., yield to maturity) of 6%. The market value of the position is $13,467,220 (134.6722% × $10 million). Since the manager owns the bond, she is concerned with a rise in yield since this will decrease the market value of the position. To assess the exposure to a rise in market yields, the manager decides to look at how the value of the bond will change if yields change instantaneously for the following three scenarios: (1) 50 basis point increase, (2) 100 basis point increase, and (3) 200 basis point increase. This means that the manager wants to assess what will happen to the bond position if the yield on the bond increases from 6% to (1) 6.5%, (2) 7%, and (3) 8%. Because this is an option-free bond, valuation is straightforward. In the examples that follow, we will use one yield to discount each of the cash flows. In other words, to simplify the calculations, we will assume a flat yield curve (even though that assumption doesn't fit the examples perfectly). The price of this bond per $100 par value and the market value of the $10 million par position is shown in Exhibit 1. Also shown is the new market value and the percentage change in market value. In the case of a portfolio, each bond is valued for a given scenario and then the total value of the portfolio is computed for a given scenario. For example, suppose that a manager has a portfolio with the following two option-free bonds: (1) 6% coupon 5-year bond and (2) 9% coupon 20-year bond. For the shorter term bond, $5 million of par value is owned and the price is 104.3760 for a yield of 5%. For the longer term bond, $10 million of par value is owned and the price is 134.6722 for a yield of 6%. Suppose that the

EXHIBIT 1 Illustration of Full Valuation Approach to Assess the Interest Rate Risk of a Bond Position for Three Scenarios

Current bond position: 9% coupon 20-year bond (option-free)
Price: 134.6722
Yield to maturity: 6%
Par value owned: $10 million
Market value of position: $13,467,220.00

Scenario	Yield change (bp)	New yield	New price	New market value ($)	Percentage change in market value (%)
1	50	6.5%	127.7606	12,776,050	−5.13%
2	100	7.0%	121.3551	12,135,510	−9.89%
3	200	8.0%	109.8964	10,989,640	−18.40%

manager wants to assess the interest rate risk of this portfolio for a 50, 100, and 200 basis point increase in interest rates assuming both the 5-year yield and 20-year yield change by the same number of basis points.

Exhibit 2 shows the interest rate risk exposure. Panel **a** of the exhibit shows the market value of the 5-year bond for the three scenarios. Panel **b** does the same for the 20-year bond. Panel **c** shows the total market value of the two-bond portfolio and the percentage change in the market value for the three scenarios.

EXHIBIT 2 Illustration of Full Valuation Approach to Assess the Interest Rate Risk of a Two Bond Portfolio (Option-Free) for Three Scenarios Assuming a Parallel Shift in the Yield Curve

Panel a

Bond 1: 6% coupon 5-year bond Par value: $5,000,000
Initial price: 104.3760 Initial market value: $5,218,800
Yield: 5%

Scenario	Yield change (bp)	New yield	New price	New market value ($)
1	50	5.5%	102.1600	5,108,000
2	100	6.0%	100.0000	5,000,000
3	200	7.0%	95.8417	4,792,085

Panel b

Bond 2: 9% coupon 20-year bond Par value: $10,000,000
Initial price: 134.6722 Initial market value: $13,467,220
Yield: 6%

Scenario	Yield change (bp)	New yield	New price	New market value ($)
1	50	6.5%	127.7605	12,776,050
2	100	7.0%	121.3551	12,135,510
3	200	8.0%	109.8964	10,989,640

Panel c

Initial Portfolio Market value: $18,686,020.00

Scenario	Yield change (bp)	Market value of Bond 1 ($)	Market value of Bond 2 ($)	Market value of Portfolio ($)	Percentage change in market value (%)
1	50	5,108,000	12,776,050	17,884,020	−4.29%
2	100	5,000,000	12,135,510	17,135,510	−8.30%
3	200	4,792,085	10,989,640	15,781,725	−15.54%

In the illustration in Exhibit 2, it is assumed that both the 5-year and the 20-year yields changed by the same number of basis points. The full valuation approach can also handle scenarios where the yield curve does not change in a parallel fashion. Exhibit 3 illustrates this for our portfolio that includes the 5-year and 20-year bonds. The scenario analyzed is a yield curve shift combined with shifts in the level of yields. In the illustration in Exhibit 3, the following yield changes for the 5-year and 20-year yields are assumed:

Scenario	Change in 5-year rate (bp)	Change in 20-year rate (bp)
1	50	10
2	100	50
3	200	100

The last panel in Exhibit 3 shows how the market value of the portfolio changes for each scenario.

EXHIBIT 3 Illustration of Full Valuation Approach to Assess the Interest Rate Risk of a Two Bond Portfolio (Option-Free) for Three Scenarios Assuming a Nonparallel Shift in the Yield Curve

Panel a

Bond 1: 6% coupon 5-year bond
Par value: $5,000,000
Initial price: 104.3760
Initial market value: $5,218,800
Yield: 5%

Scenario	Yield change (bp)	New yield	New price	New market value ($)
1	50	5.5%	102.1600	5,108,000
2	100	6.0%	100.0000	5,000,000
3	200	7.0%	95.8417	4,792,085

Panel b

Bond 2: 9% coupon 20-year bond
Par value: $10,000,000
Initial price: 134.6722
Initial market value: $13,467,220
Yield: 6%

Scenario	Yield change (bp)	New yield	New price	New market value ($)
1	10	6.1%	133.2472	13,324,720
2	50	6.5%	127.7605	12,776,050
3	100	7.0%	121.3551	12,135,510

Panel c

Initial Portfolio Market value: $18,686,020.00

	Market value of			Percentage change in
Scenario	Bond 1 ($)	Bond 2 ($)	Portfolio ($)	market value (%)
1	5,108,000	13,324,720	18,432,720	−1.36%
2	5,000,000	12,776,050	17,776,050	−4.87%
3	4,792,085	12,135,510	16,927,595	−9.41%

The full valuation approach seems straightforward. If one has a good valuation model, assessing how the value of a portfolio or individual bond will change for different scenarios for parallel and nonparallel yield curve shifts measures the interest rate risk of a portfolio.

A common question that often arises when using the full valuation approach is which scenarios should be evaluated to assess interest rate risk exposure. For some regulated entities, there are specified scenarios established by regulators. For example, it is common for regulators of depository institutions to require entities to determine the impact on the value of their bond portfolio for a 100, 200, and 300 basis point instantaneous change in interest rates (up and down). (Regulators tend to refer to this as ''simulating'' interest rate scenarios rather than scenario analysis.) Risk managers and highly leveraged investors such as hedge funds tend to look at extreme scenarios to assess exposure to interest rate changes. This practice is referred to as stress testing.

Of course, in assessing how changes in the yield curve can affect the exposure of a portfolio, there are an infinite number of scenarios that can be evaluated. The state-of-the-art technology involves using a complex statistical procedure to determine a likely set of yield curve shift scenarios from historical data.

We can use the full valuation approach to assess the exposure of a bond or portfolio to interest rate changes to evaluate any scenario, assuming—and this must be repeated continuously—that the manager has a good valuation model to estimate what the price of the bonds will be in each interest rate scenario. However, we are not stopping here. The reason is that the full valuation process can be very time consuming. This is particularly true if the portfolio has a large number of bonds, even if a minority of those bonds are complex (i.e., have embedded options). While the full valuation approach is the recommended method, managers want one simple measure that they can use to get an idea of how bond prices will change if rates change in a parallel fashion, rather than having to revalue an entire portfolio. This measure is called — duration. In order to understand this measure, we should first understand the concept of price volatility of bonds.

PRICE VOLATILITY CHARACTERISTICS OF BONDS

Here are the characteristics of a bond that affect its price volatility:

(1) maturity, (2) coupon rate, and (3) presence of embedded options. We also explained how the level of yields affects price volatility. In this section, we will take a closer look at the price volatility of bonds.

Price Volatility Characteristics of Option-Free Bonds

Let's begin by focusing on option-free bonds (i.e., bonds that do not have embedded options). A fundamental characteristic of an option-free bond is that the price of the bond changes in the opposite direction to a change in the bond's yield. Exhibit 4 illustrates this property for four hypothetical bonds assuming a par value of $100.

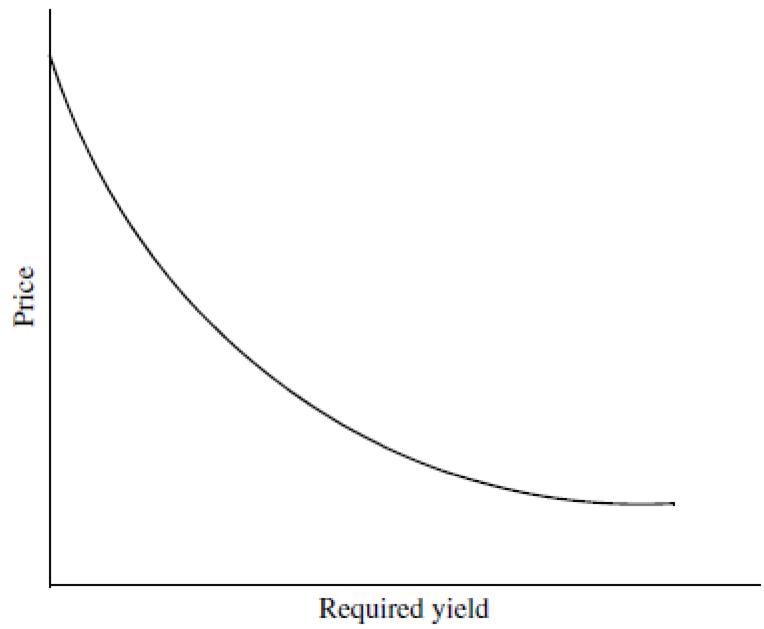
EXHIBIT 4 Price/Yield Relationship for Four Hypothetical Option-Free Bonds

	Price ($)			
Yield (%)	6%/5 year	6%/20 year	9%/5 year	9%/20 year
4.00	108.9826	127.3555	122.4565	168.3887
5.00	104.3760	112.5514	117.5041	150.2056
5.50	102.1600	106.0195	115.1201	142.1367
5.90	100.4276	101.1651	113.2556	136.1193
5.99	100.0427	100.1157	112.8412	134.8159
6.00	100.0000	100.0000	112.7953	134.6722
6.01	99.9574	99.8845	112.7494	134.5287
6.10	99.5746	98.8535	112.3373	133.2472
6.50	97.8944	94.4479	110.5280	127.7605
7.00	95.8417	89.3225	108.3166	121.3551
8.00	91.8891	80.2072	104.0554	109.8964

When the price/yield relationship for any option-free bond is graphed, it exhibits the shape shown in Exhibit 5. Notice that as the yield increases, the price of an option-free bond declines. However, this relationship is not linear (i.e., not a straight line relationship). The shape of the price/yield relationship for any option-free bond is referred to as convex. This price/yield relationship reflects an instantaneous change in the required yield.

The price sensitivity of a bond to changes in the yield can be measured in terms of the dollar price change or the percentage price change.

EXHIBIT 5 Price/Yield Relationship for a Hypothetical Option-Free Bond



Maximum price = sum of undiscounted cash flows

Exhibit 6 uses the four hypothetical bonds in Exhibit 4 to show the percentage change in each bond's price for various changes in yield, assuming that the initial yield for all four bonds is 6%. An examination of Exhibit 6 reveals the following properties concerning the price volatility of an option-free bond:

Property 1: Although the price moves in the opposite direction from the change in yield, the percentage price change is not the same for all bonds.

EXHIBIT 6 Instantaneous Percentage Price Change for Four Hypothetical Bonds (Initial yield for all four bonds is 6%)

	Percentage Price Change			
New Yield (%)	6%/5 year	6%/20 year	9%/5 year	9%/20 year
4.00	8.98	27.36	8.57	25.04
5.00	4.38	12.55	4.17	11.53
5.50	2.16	6.02	2.06	5.54
5.90	0.43	1.17	0.41	1.07
5.99	0.04	0.12	0.04	0.11
6.01	−0.04	−0.12	−0.04	−0.11
6.10	−0.43	−1.15	−0.41	−1.06
6.50	−2.11	−5.55	−2.01	−5.13
7.00	−4.16	−10.68	−3.97	−9.89
8.00	−8.11	−19.79	−7.75	−18.40

Property 2: For small changes in the yield, the percentage price change for a given bond is roughly the same, whether the yield increases or decreases.

Property 3: For large changes in yield, the percentage price change is not the same for an increase in yield as it is for a decrease in yield.

Property 4: For a given large change in yield, the percentage price increase is greater than the percentage price decrease.

While the properties are expressed in terms of percentage price change, they also hold for dollar price changes.

An explanation for these last two properties of bond price volatility lies in the convex shape of the price/yield relationship. Exhibit 7 illustrates this. The following notation is used

in the exhibit:

Y = initial yield

Y1 = lower yield

Y2 = higher yield

P = initial price

P1 = price at lower yield Y1

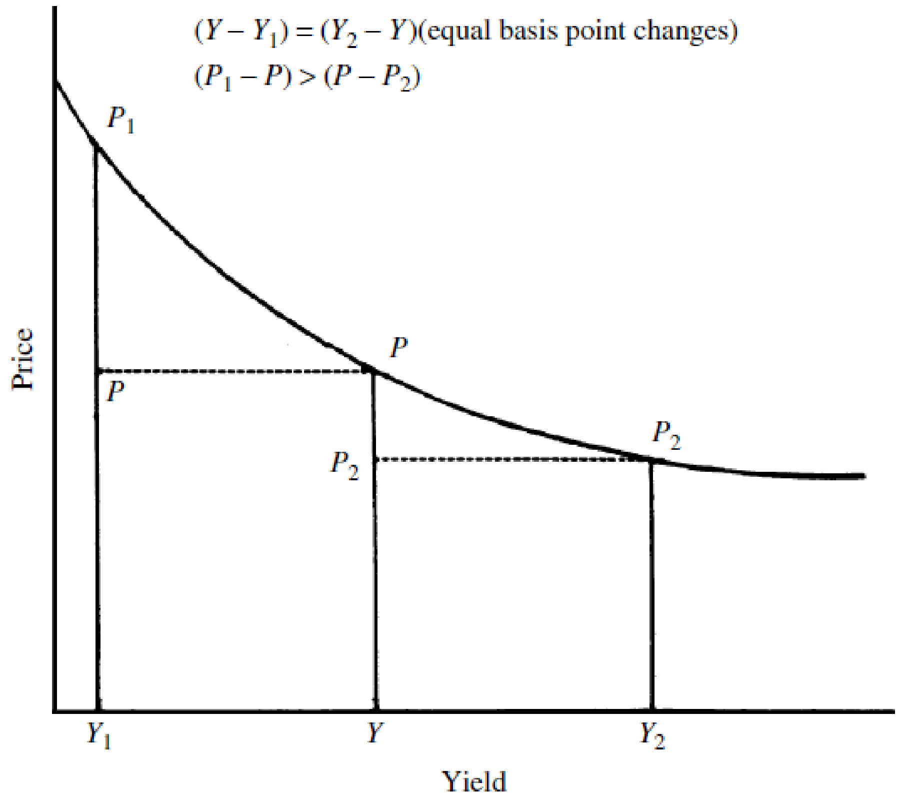
P2 = price at higher yield Y2

What was done in the exhibit was to change the initial yield (Y) up and down by the same number of basis points. That is, in Exhibit 7, the yield is decreased from Y to Y1 and increased from Y to Y2 such that the change is the same:

Y – Y1 = Y2 – Y

Also, the change in yield is a large number of basis points. The vertical distance from the horizontal axis (the yield) to the intercept on the graph shows the price. The change in the initial price (P) when the yield declines from Y to Y1 is equal to the difference between the new price (P1) and the initial price (P). That is, change in price when yield decreases = P1 – P

EXHIBIT 7 Graphical Illustration of Properties 3 and 4 for an Option-Free Bond



The change in the initial price (P) when the yield increases from Y to Y2 is equal to the difference between the new price (P2) and the initial price (P). That is,

change in price when yield increases = P2 – P

As can be seen in the exhibit, the change in price when yield decreases is not equal to the change in price when yield increases by the same number of basis points. That is,

(P1 – P) <> (P2 – P)

This is what Property 3 states.

A comparison of the price change shows that the change in price when yield decreases is greater than the change in price when yield increases. That is,

P1 − P > P2 − P

This is Property 4.

The implication of Property 4 is that if an investor owns a bond, the capital gain that will be realized if the yield decreases is greater than the capital loss that will be realized if the yield increases by the same number of basis points. For an investor who is short a bond (i.e., sold a bond not owned), the reverse is true: the potential capital loss is greater than the potential capital gain if the yield changes by a given number of basis points.

The convexity of the price/yield relationship impacts Property 4. Exhibit 8 shows a less convex price/yield relationship than Exhibit 7. That is, the price/yield relationship in Exhibit 8 is less bowed than the price/yield relationship in Exhibit 7. Because of the difference in the convexities, look at what happens when the yield increases and decreases by the same number of basis points and the yield change is a large number of basis points. We use the same notation in Exhibits 8 and 9 as in Exhibit 7. Notice that while the price gain when the yield decreases is greater than the price decline when the yield increases, the gain is not much greater than the loss. In contrast, Exhibit 9 has much greater convexity than the bonds in Exhibits 7 and 8 and the price gain is significantly greater than the loss for the bonds depicted in Exhibits 7 and 8.

EXHIBIT 8 Impact of Convexity on Property 4: Less Convex Bond

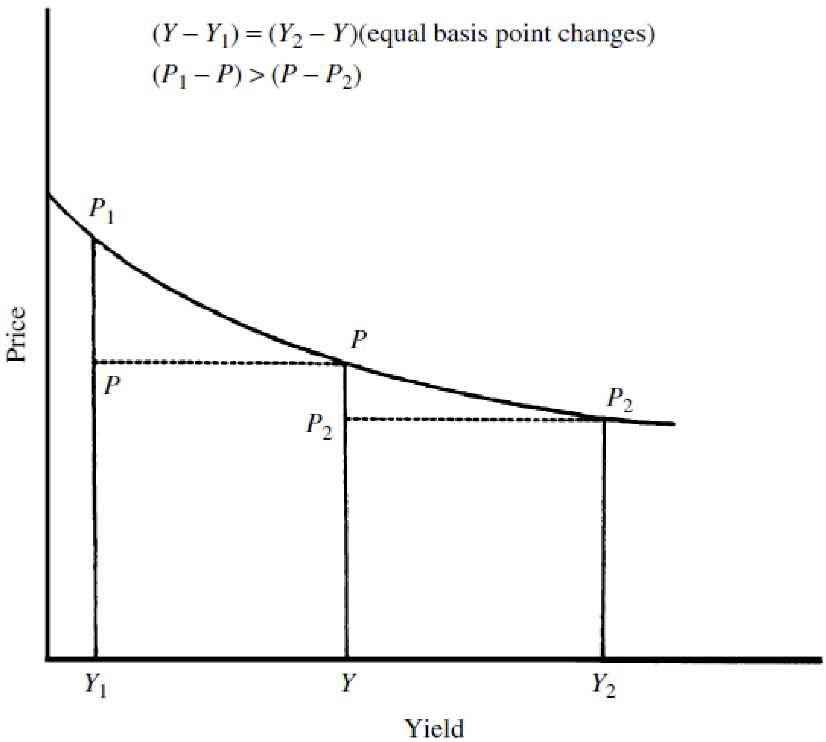
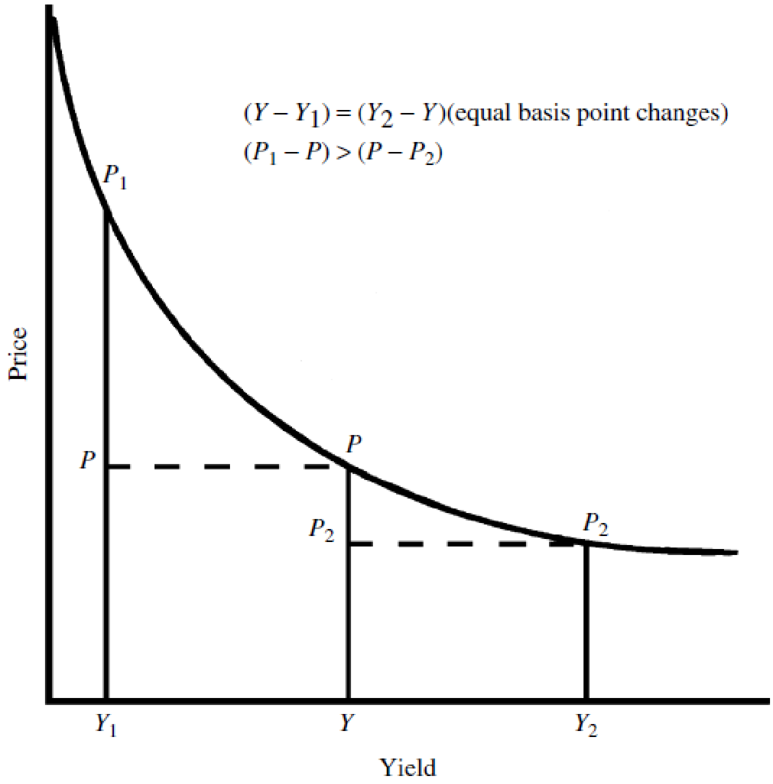


EXHIBIT 9 Impact of Convexity on Property 4: Highly Convex Bond



Price Volatility of Bonds with Embedded Options

Now let's turn to the price volatility of bonds with embedded options. As explained in previous chapters, the price of a bond with an embedded option is comprised of two components. The first is the value of the same bond if it had no embedded option (that is, the price if the bond is option free). The second component is the value of the embedded option. In other words, the value of a bond with embedded options is equal to the value of an option-free bond plus or minus the value of embedded options.

The two most common types of embedded options are call (or prepay) options and put options. As interest rates in the market decline, the issuer may call or prepay the debt obligation prior to the scheduled principal payment date. The other type of option is a put option. This option gives the investor the right to require the issuer to purchase the bond at a specified price. Below we will examine the price/yield relationship for bonds with both types of embedded options (calls and puts) and implications for price volatility.

Bonds with Call and Prepay Options

In the discussion below, we will refer to a bond that may be called or is pre-payable as a callable bond. Exhibit 10 shows the price/yield relationship for an option-free bond and a callable bond. The convex curve given by **a–a** is the price/yield relationship for an option-free bond. The unusual shaped curve denoted by **a–b** in the exhibit is the price/yield relationship for the callable bond.

The reason for the price/yield relationship for a callable bond is as follows. When the prevailing market yield for comparable bonds is higher than the coupon rate on the callable bond, it is unlikely that the issuer will call the issue. For example, if the coupon rate on a bond is 7% and the prevailing market yield on comparable bonds is 12%, it is highly unlikely that the issuer will call a 7% coupon bond so that it can issue a 12% coupon bond. Since the bond is unlikely to be called, the callable bond will have a similar price/yield relationship to an otherwise comparable option-free bond. Consequently, the callable bond will be valued as if it is an option-free bond. However, since there is still some value to the call option, the bond won't trade exactly like an option-free bond.

As yields in the market decline, the concern is that the issuer will call the bond. The issuer won't necessarily exercise the call option as soon as the market yield drops below the coupon rate. Yet, the value of the embedded call option increases as yields approach the coupon rate from higher yield levels. For example, if the coupon rate on a bond is 7% and the market yield declines to 7.5%, the issuer will most likely not call the issue. However, market yields are now at a level at which the investor is concerned that the issue may eventually be called if market yields decline further. Cast in terms of the value of the embedded call option, that option becomes more valuable to the issuer and therefore it reduces the price relative to an otherwise comparable option-free bond.3 In Exhibit 10, the value of the embedded call option at a given yield can be measured by the difference between the price of an option-free bond (the price shown on the curve a–a) and the price on the curve a–b. Notice that at low yield levels (below $y*$ on the horizontal axis), the value of the embedded call option is high.

Using the information in Exhibit 10, let's compare the price volatility of a callable bond to that of an option-free bond. Exhibit 11 focuses on the portion of the price/yield relationship for the callable bond where the two curves in Exhibit 10 depart (segment **b–b** in Exhibit 10).

We know from our earlier discussion that for a large change in yield, the price of an option-free bond increases by more than it decreases (Property 4 above). Is that what happens for a callable bond in the region of the price/yield relationship shown in Exhibit 11? No, it is not. In fact, as can be seen in the exhibit, the opposite is true! That is, for a given large change in yield, the price appreciation is less than the price decline.

This very important characteristic of a callable bond—that its price appreciation is less than its price decline when rates change by a large number of basis points—is referred to as negative convexity.4 But notice from Exhibit 10 that callable bonds don't exhibit this characteristic at every yield level. When yields are high (relative to the issue's coupon rate), the bond exhibits the same price/yield relationship as an option-free bond; therefore at high yield levels it also has the characteristic that the gain is greater than the loss. Because market participants have referred to the shape of the price/yield relationship shown in Exhibit 11 as negative convexity, market participants refer to the relationship for an option-free bond as positive convexity. Consequently, a callable bond exhibits negative convexity at low yield levels and positive convexity at high yield levels. This is depicted in Exhibit 12.

EXHIBIT 10 Price/Yield Relationship for a Callable Bond and an Option-Free Bond

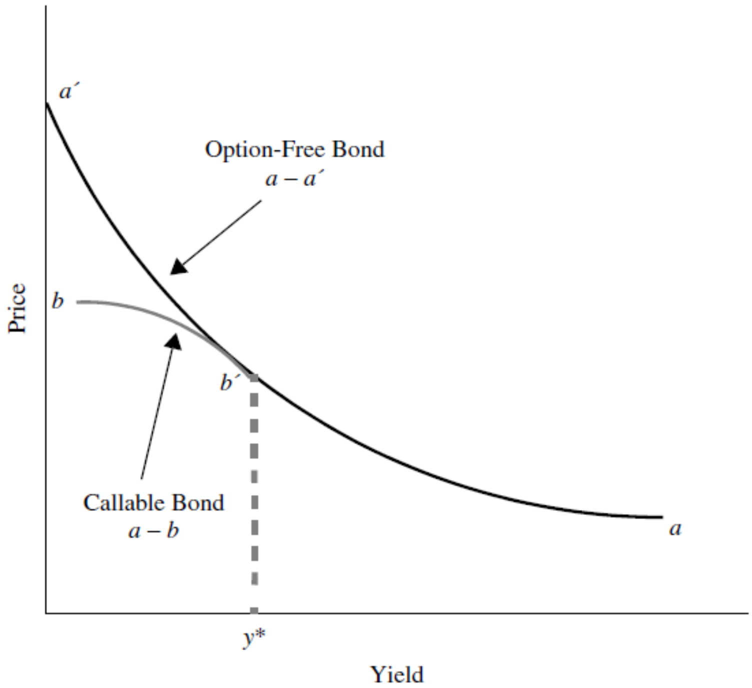
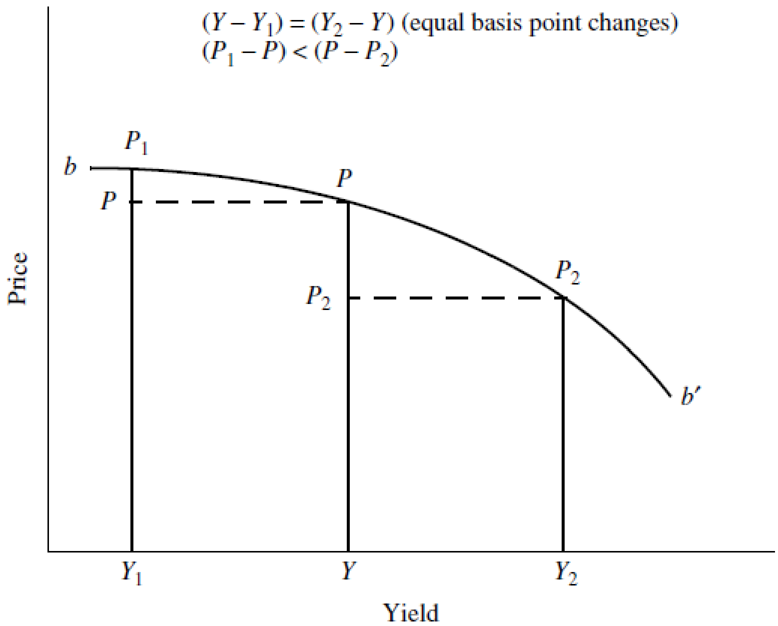
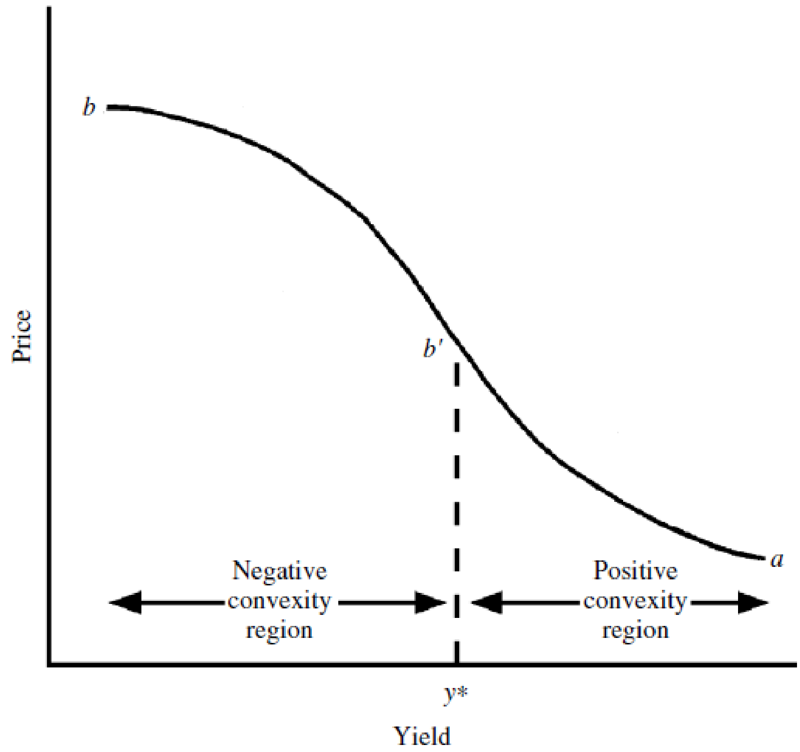


EXHIBIT 11 Negative Convexity Region of the Price/Yield Relationship for a Callable Bond



As can be seen from the exhibits, when a bond exhibits negative convexity, the bond compresses in price as rates decline. That is, at a certain yield level there is very little price appreciation when rates decline. When a bond enters this region, the bond is said to exhibit ‘‘price compression.’’

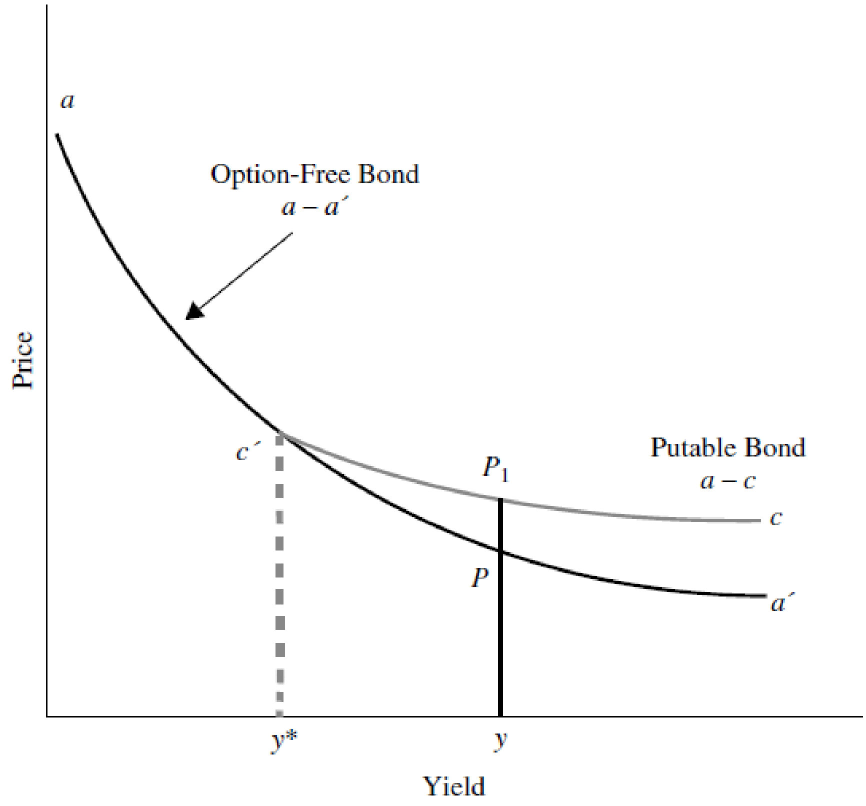
EXHIBIT 12 Negative and Positive Convexity Exhibited by a Callable Bond



Bonds with Embedded Put Options

Putable bonds may be redeemed by the bondholder on the dates and at the put price specified in the indenture. Typically, the put price is par value. The advantage to the investor is that if yields rise such that the bond’s value falls below the put price, the investor will exercise the put option. If the put price is par value, this means that if market yields rise above the coupon rate, the bond’s value will fall below par and the investor will then exercise the put option.

EXHIBIT 13 Price/Yield Relationship for a Putable Bond and an Option-Free Bond



The value of a putable bond is equal to the value of an option-free bond plus the value of the put option. Thus, the difference between the value of a putable bond and the value of an otherwise comparable option-free bond is the value of the embedded put option. This can be seen in Exhibit 13 which shows the price/yield relationship for a putable bond is the curve **a–c** and for an option-free bond is the curve **a–a**.

At low yield levels (low relative to the issue's coupon rate), the price of the putable bond is basically the same as the price of the option-free bond because the value of the put option is small. As rates rise, the price of the putable bond declines, but the price decline is less than that for an option-free bond. The divergence in the price of the putable bond and an otherwise comparable option-free bond at a given yield level (y) is the value of the put option (P1 –P).

When yields rise to a level where the bond's price would fall below the put price, the price at these levels is the put price.

DURATION

With the background about the price volatility characteristics of a bond, we can now turn to an alternate approach to full valuation: the duration/convexity approach. Recall from the earlier section that **duration is a measure of the approximate price sensitivity of a bond to interest rate changes.**

More specifically, it is the approximate percentage change in price for a 100 basis point change in rates. We will see in this section that duration is the first (linear) approximation of the percentage price change. To improve the approximation provided by duration, an adjustment for ‘‘convexity’’ can be made. Hence**, using duration combined with convexity to estimate the percentage price change of a bond caused by changes in interest rates is called the duration/convexity approach.**

Calculating Duration

The duration of a bond is estimated as follows:

$$\frac{\text{price if yields decline} - \text{price if yields rise}}{2(\text{initial price})(\text{change in yield in decimal})}$$

If we let

$$\begin{aligned} \Delta y &= \text{change in yield in decimal} \\ V_0 &= \text{initial price} \\ V- &= \text{price if yields decline by} \Delta y \\ V_+ &= \text{price if yields increase by} \Delta y \end{aligned}$$

then duration can be expressed as

$$\text{duration} = \frac{V_- - V_+}{2(V_0)(\Delta_y)}$$

For example, consider a 9% coupon 20-year option-free bond selling at 134.6722 to yield 6% (see Exhibit 4). Let’s change (i.e., shock) the yield down and up by 20 basis points and determine what the new prices will be for the numerator. If the yield is decreased by 20 basis points from 6.0% to 5.8%, the price would increase to 137.5888. If the yield increases by 20 basis points, the price would decrease to 131.8439. Thus,

Δy = 0.002

V0 = 134.6722

V− = 137.5888

V+ = 131.8439

Then, duration = 137.5888 − 131.8439

2 × (134.6722) × (0.002) = 10.66

Duration is interpreted as the approximate percentage change in price for a 100 basis point change in rates. Consequently, the duration of 10.66 means that the approximate change in price for this bond is 10.66% for a 100 basis point change in rates. A common question asked about this interpretation of duration is the consistency between the yield change that is used to compute duration using equation (1) and the interpretation of duration. For example, recall that in computing the duration of the 9% coupon 20-year bond, we used a 20 basis point yield change to obtain the two prices to use in the numerator of equation (1). Yet, we interpret the duration computed as the approximate percentage price change for a 100 basis point change in yield. The reason is that regardless of the yield change used to estimate duration in equation (1), the interpretation is the same. If we used a 25 basis point change in yield to compute the prices used in the numerator of equation (1), the resulting duration is interpreted as the approximate percentage price change for a 100 basis point change in yield. Later we will use different changes in yield to illustrate the sensitivity of the computed duration.

Approximating the Percentage Price Change Using Duration

Earlier we explained how to approximate the percentage price change for a given change in yield and a given duration. Here we will express the process using the following formula:

approximate percentage price change = −duration × Δy × 100 **(2)**

where **Δy** is the yield change (in decimal) for which the estimated percentage price change is sought. The reason for the negative sign on the right-hand side of equation (2) is due to the inverse relationship between price change and yield change (e.g., as yields increase, bond prices decrease). The following two examples illustrate how to use duration to estimate a bond's price change.

Example #1: small change in basis point yield. For example, consider the 9% 20-year bond trading at 134.6722 whose duration we just showed is 10.66. The approximate percentage price change for a 10 basis point increase in yield (i.e., Δy = +0.001) is:

approximate percentage price change = −10.66 × (+0.001) × 100 = −1.066%

How good is this approximation? The actual percentage price change is −1.06% (as shown in Exhibit 6 when yield increases to 6.10%). Duration, in this case, did an excellent job in estimating the percentage price change.

We would come to the same conclusion if we used duration to estimate the percentage price change if the yield declined by 10 basis points (i.e., Δy = −0.001). In this case, the approximate percentage price change would be +1.066% (i.e., the direction of the estimated price change is the reverse but the magnitude of the change is the same). The exhibit below shows that the actual percentage price change is +1.07%.

In terms of estimating the new price, let's see how duration performed. The initial price is 134.6722. For a 10 basis point increase in yield, duration estimates that the price will decline by 1.066%. Thus, the price will decline to 133.2366 (found by multiplying 134.6722 by one minus 0.01066). The actual price if the yield increases by 10 basis points is 133.2472. Thus, the price estimated using duration is close to the actual price.

For a 10 basis point decrease in yield, the actual price is 136.1193 and the estimated price using duration is 136.1078 (a price increase of 1.066%). Consequently, the new price estimated by duration is close to the actual price for a 10 basis point change in yield.

Example #2: large change in basis point yield. Let's look at how well duration does in estimating the percentage price change if the yield increases by 200 basis points instead of 10 basis points. In this case, y is equal to +0.02. Substituting into equation (2), we have approximate percentage price change = −10.66 × (+0.02)×100 = −21.32%

How good is this estimate? From the exhibit below, we see that the actual percentage price change when the yield increases by 200 basis points to 8% is −18.40%. Thus, the estimate is not as accurate as when we used duration to approximate the percentage price change for a change in yield of only 10 basis points. If we use duration to approximate the percentage price change when the yield decreases by 200 basis points, the approximate percentage price change in this scenario is +21.32%. The actual percentage price change as shown in the exhibit is +25.04%.

Let's look at the use of duration in terms of estimating the new price. Since the initial price is 134.6722 and a 200 basis point increase in yield will decrease the price by 21.32%, the estimated new price using duration is 105.9601 (found by multiplying 134.6722 by one minus 0.2132). The actual price if the yield is 8% is 109.8964. Consequently, the estimate is not as accurate as the estimate for a 10 basis point change in yield. The estimated new price using duration for a 200 basis point decrease in yield is 163.3843 compared to the actual price (from the exhibit below) of 168.3887. Once again, the estimation of the price using duration is not as accurate as for a 10 basis point change. Notice that whether the yield is increased or decreased by 200 basis points, duration underestimates what the new price will be. We will see why shortly.

Summary. Let's summarize what we found in our application of duration to approximate the percentage price change:

Yield change	Initial	New price		Percent price change		
(bp)	price	Based on duration	Actual	Based on duration	Actual	Comment
+10	134.6722	133.2366	133.2472	−1.066	−1.06	estimated price close to new price
−10	134.6722	136.1078	136.1193	+1.066	+1.07	estimated price close to new price
+200	134.6722	105.9601	109.8964	−21.320	−18.40	underestimates new price
−200	134.6722	163.3843	168.3887	+21.320	+25.04	underestimates new price

Should any of this be a surprise to you? No, not after evaluating equation (2) in terms of the properties for the price/yield relationship discussed in that section. Look again at equation (2). Notice that whether the change in yield is an increase or a decrease, the approximate percentage price change will be the same except that the sign is reversed. This violates Property 3 and Property 4 with respect to the price volatility of option-free bonds when yields change. Recall that Property 3 states that the percentage price change will not be the same for a large increase and decrease in yield by the same number of basis points. Property 4 states the percentage price increase is greater than the percentage price decrease. These are two reasons why the estimate is inaccurate for a 200 basis point yield change.

Why did the duration estimate of the price change do a good job for a small change in yield of 10 basis points? Recall from Property 2 that the percentage price change will be approximately the same whether there is an increase or decrease in yield by a small number of basis points. We can also explain these results in terms of the graph of the price/yield relationship.

Graphical Depiction of Using Duration to Estimate Price Changes

We used the graph of the price/yield relationship to demonstrate the price volatility properties of bonds. We can also use graphs to illustrate what we observed in our examples about how duration estimates the percentage price change, as well as some other noteworthy points.

The shape of the price/yield relationship for an option-free bond is convex. Exhibit 14 shows this relationship. In the exhibit, a tangent line is drawn to the price/yield relationship at yield $y*$. (For those unfamiliar with the concept of a tangent line, it is a straight line that just touches a curve at one point within a relevant (local) range. In Exhibit 14, the tangent line touches the curve at the point where the yield is equal to $y*$ and the price is equal to $p*$.). The tangent line is used to estimate the new price if the yield changes. If we draw a vertical line from any yield (on the horizontal axis), as in Exhibit 14, the distance between the horizontal axis and the tangent line represents the price approximated by using duration starting with the initial yield $y*$.

Now how is the tangent line related to duration? Given an initial price and a specific yield change, the tangent line tells us the approximate new price of a bond. The approximate percentage price change can then be computed for this change in yield. But this is precisely what duration [using equation (2)] gives us: the approximate percentage price change for a given change in yield. Thus, using the tangent line, one obtains the same approximate percentage price change as using equation (2).

This helps us understand why duration did an effective job of estimating the percentage price change, or equivalently the new price, when the yield changes by a small number of basis points. Look at Exhibit 15. Notice that for a small change in yield, the tangent line does not depart much from the price/yield relationship. Hence, when the yield changes up or down by 10 basis points, the tangent line does a good job of estimating the new price, as we found in our earlier numerical illustration.

Exhibit 15 shows what happens to the estimate using the tangent line when the yield changes by a large number of basis points. Notice that the error in the estimate gets larger the further one moves from the initial yield. The estimate is less accurate the more convex the bond as illustrated in Exhibit 16.

EXHIBIT 14 Price/Yield Relationship for an Option-Free Bond with a Tangent Line

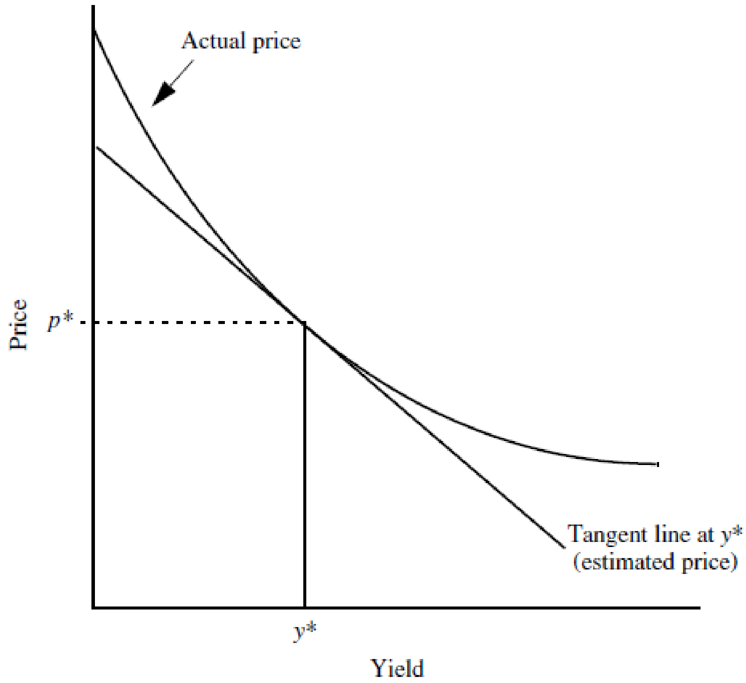


EXHIBIT 15 Estimating the New Price Using a Tangent Line

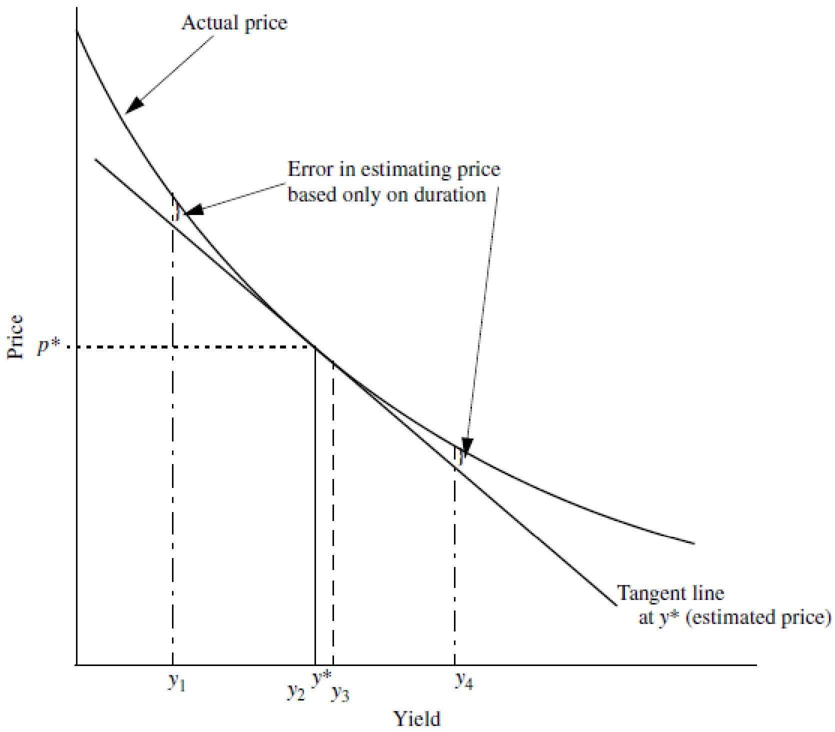
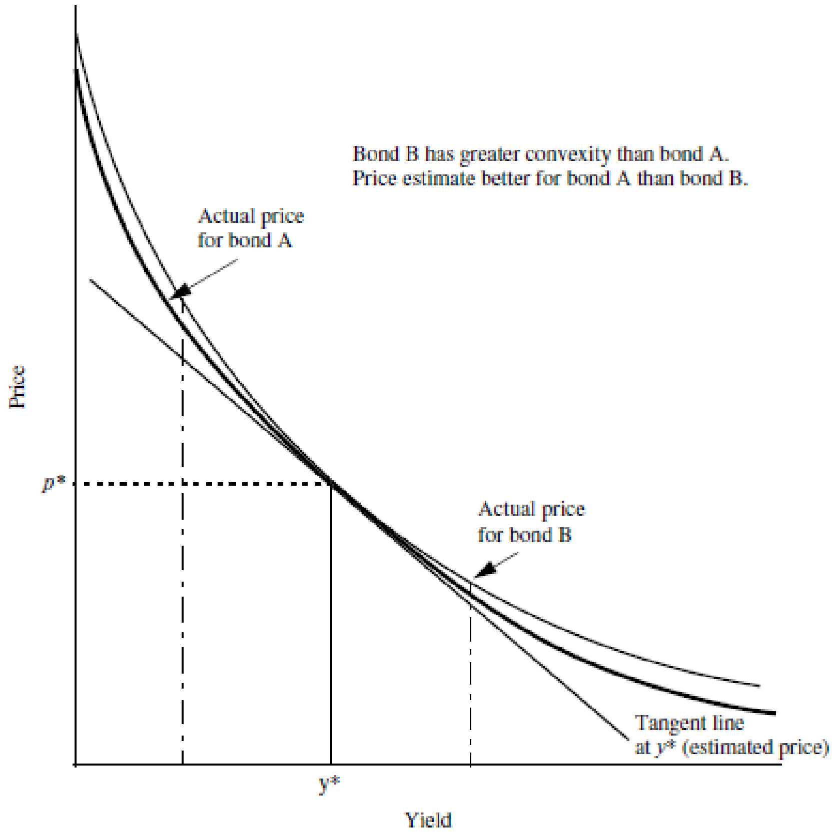


EXHIBIT 16 Estimating the New Price for a Large Yield Change for Bonds with Different Convexities



Also note that, regardless of the magnitude of the yield change, the tangent line always underestimates what the new price will be for an option-free bond because the tangent line is below the price/yield relationship. This explains why we found in our illustration that when using duration, we underestimated what the actual price will be.

The results reported in Exhibit 17 are for option-free bonds. When we deal with more complicated securities, small rate shocks that do not reflect the types of rate changes that may occur in the market do not permit the determination of how prices can change. This is because expected cash flows may change when dealing with bonds with embedded options. In comparison, if large rate shocks are used, we encounter the asymmetry caused by convexity.

Moreover, large rate shocks may cause dramatic changes in the expected cash flows for bonds with embedded options that may be far different from how the expected cash flows will change for smaller rate shocks.

There is another potential problem with using small rate shocks for complicated securities. The prices that are inserted into the duration formula as given by equation (2) are derived from a valuation model. The duration measure depends crucially on the valuation model. If the rate shock is small and the valuation model used to obtain the prices for equation (1) is poor, dividing poor price estimates by a small shock in rates (in the denominator) will have a significant effect on the duration estimate.

Rate Shocks and Duration Estimate

In calculating duration using equation (1), it is necessary to shock interest rates (yields) up and down by the same number of basis points to obtain the values for V− and V+. In our illustration, 20 basis points was arbitrarily selected. But how large should the shock be? That is, how many basis points should be used to shock the rate?

In Exhibit 17, the duration estimates for our four hypothetical bonds using equation (1) for rate shocks of 1 basis point to 200 basis points are reported. The duration estimates for the two 5-year bonds are not affected by the size of the shock. The two 5-year bonds are less convex than the two 20-year bonds. But even for the two 20-year bonds, for the size of the shocks reported in Exhibit 17, the duration estimates are not materially affected by the greater convexity.

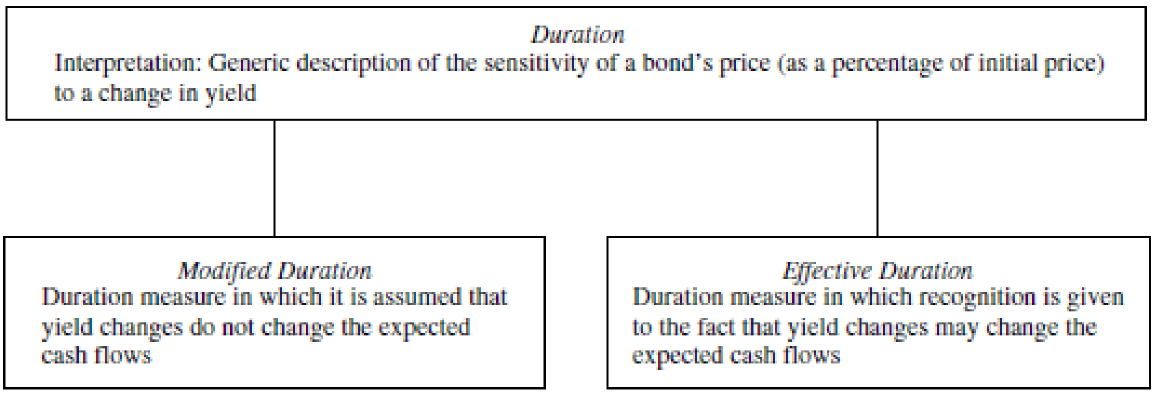
What is done in practice by dealers and vendors of analytical systems? Each system developer uses rate shocks that they have found to be realistic based on historical rate changes.

EXHIBIT 17 Duration Estimates for Different Rate Shocks

Assumption: Initial yield is 6%

Bond	1 bp	10 bps	20 bps	50 bps	100 bps	150 bps	200 bps
6% 5 year	4.27	4.27	4.27	4.27	4.27	4.27	4.27
6% 20 year	11.56	11.56	11.56	11.57	11.61	11.69	11.79
9% 5 year	4.07	4.07	4.07	4.07	4.07	4.08	4.08
9% 20 year	10.66	10.66	10.66	10.67	10.71	10.77	10.86

EXHIBIT 18 Modified Duration versus Effective Duration



Modified Duration versus Effective Duration

One form of duration that is cited by practitioners is modified duration. Modified duration is the approximate percentage change in a bond's price for a 100 basis point change in yield assuming that the bond's expected cash flows do not change when the yield changes. What this means is that in calculating the values of V_- and V_+ in equation (1), the same cash flows used to calculate V_0 are used. Therefore, the change in the bond's price when the yield is changed is due solely to discounting cash flows at the new yield level.

The assumption that the cash flows will not change when the yield is changed makes sense for option-free bonds such as non-callable Treasury securities. This is because the payments made by the U.S. Department of the Treasury to holders of its obligations do not change when interest rates change. However, the same cannot be said for bonds with embedded options (i.e., callable and putable bonds and mortgage-backed securities). For these securities, a change in yield may significantly alter the expected cash flows.

Earlier we showed the price/yield relationship for callable and pre-payable bonds. Failure to recognize how changes in yield can alter the expected cash flows will produce two values used in the numerator of equation (1) that are not good estimates of how the price will actually change. The duration is then not a good number to use to estimate how the price will change.

Some valuation models for bonds with embedded options take into account how changes in yield will affect the expected cash flows. Thus, when V_- and V_+ are the values produced from these valuation models, the resulting duration takes into account both the discounting at different interest rates and how the expected cash flows may change. When duration is calculated in this manner, it is referred to as effective duration or option-adjusted duration (Lehman Brothers refers to this measure in some of its publications as adjusted duration).

Exhibit 18 summarizes the distinction between modified duration and effective duration. The difference between modified duration and effective duration for bonds with embedded options can be quite dramatic. For example, a callable bond could have a modified duration of 5 but an effective duration of only 3. For certain collateralized mortgage obligations, the modified duration could be 7 and the effective duration 20! Thus, using modified duration as a measure of the price sensitivity for a security with embedded options to changes in yield would be misleading. Effective duration is the more appropriate measure for any bond with an embedded option.

Macaulay Duration and Modified Duration

It is worth comparing the relationship between modified duration to the another duration measure, Macaulay duration. Modified duration can be written as:

$$\frac{1}{(1+\text{yield}/k)}\left[\frac{1\times \text{PVCF}_1 + 2\times \text{PVCF}_2 \ldots + n\times \text{PVCF}_n}{k\times \text{Price}}\right] \quad (3)$$

where

k = number of periods, or payments, per year (e.g., $k = 2$ for semiannual-pay bonds and $k = 12$ for monthly-pay bonds)

n = number of periods until maturity (i.e., number of years to maturity times k)

yield = yield to maturity of the bond

PVCF_t = present value of the cash flow in period t discounted at the yield to maturity where $t = 1, 2, \ldots, n$

We know that duration tells us the approximate percentage price change for a bond if the yield changes. The expression in the brackets of the modified duration formula given by equation (3) is a measure formulated in 1938 by Frederick Macaulay. This measure is popularly referred to as Macaulay duration. Thus, modified duration is commonly expressed as:

$$\text{Modified duration} = \frac{\text{Macaulay duration}}{(1+\text{yield}/k)}$$

The general formulation for duration as given by equation (1) provides a short-cut procedure for determining a bond's modified duration. Because it is easier to calculate the modified duration using the short-cut procedure, most vendors of analytical software will use equation (1) rather than equation (3) to reduce computation time.

However, modified duration is a flawed measure of a bond's price sensitivity to interest rate changes for a bond with embedded options and therefore so is Macaulay duration. The duration formula given by equation (3) misleads the user because it masks the fact that changes in the expected cash flows must be recognized for bonds with embedded options. Although equation (3) will give the same estimate of percent price change for an option-free bond as equation (1), equation (1) is still better because it acknowledges cash flows and thus value can change due to yield changes.

Interpretations of Duration

Throughout this book, the definition provided for duration is: the approximate percentage price change for a 100 basis point change in rates. That definition is the most relevant for how a manager or investor uses duration. In fact, if you understand this definition, you can easily calculate the change in a bond's value.

For example, suppose we want to know the approximate percentage change in price for a 50 basis point change in yield for our hypothetical 9% coupon 20-year bond selling for 134.6722. Since the duration is 10.66, a 100 basis point change in yield would change the price by about 10.66%. For a 50 basis point change in yield, the price will change by approximately 5.33% (= 10.66%/2). So, if the yield increases by 50 basis points, the price will decrease by about 5.33% from 134.6722 to 127.4942.

Now let's look at some other duration definitions or interpretations that appear in publications and are cited by managers in discussions with their clients.

Duration Is the ''First Derivative''

Sometimes a market participant will refer to duration as the ''first derivative of the price/yield function'' or simply the ''first derivative.'' Wow! Sounds impressive. First, ''derivative'' here has nothing to do with ''derivative instruments'' (i.e., futures, swaps, options, etc.). A derivative as used in this context is obtained by differentiating a mathematical function using calculus. There are first derivatives, second derivatives, and so on. When market participants say that duration is the first derivative, here is what they mean. The first derivative calculates the slope of a line—in this case, the slope of the tangent line in Exhibit 14. If it were possible to write a mathematical equation for a bond in closed form, the first derivative would be the result of differentiating that equation the first time. Even if you don't know how to do the process of differentiation to get the first derivative, it sounds like you are really smart since it suggests you understand calculus!

While it is a correct interpretation of duration, it is an interpretation that in no way helps us understand what the interest rate risk is of a bond. That is, it is an operationally meaningless interpretation.

Why is it an operationally meaningless interpretation? Go back to the $10 million bond position with a duration of 6. Suppose a client is concerned with the exposure of the bond to changes in interest rates. Now, tell that client the duration is 6 and that it is the first derivative of the price function for that bond. What have you told the client? Not much. In contrast, tell that client that the duration is 6 and that duration is the approximate price sensitivity of a bond to a 100 basis point change in rates and you have told the client more relevant information with respect the bond's interest rate risk.

Duration Is Some Measure of Time

When the concept of duration was originally introduced by Macaulay in 1938, he used it as a gauge of the time that the bond was outstanding. More specifically, Macaulay defined duration as the weighted average of the time to each coupon and principal payment of a bond. Subsequently, duration has too often been thought of in temporal terms, i.e., years. This is most unfortunate for two reasons.

First, in terms of dimensions, there is nothing wrong with expressing duration in terms of years because that is the proper dimension of this value. But the proper interpretation is that duration is the price volatility of a zero-coupon bond with that number of years to maturity.

So, when a manager says a bond has a duration of 4 years, it is not useful to think of this measure in terms of time, but that the bond has the price sensitivity to rate changes of a 4-year zero-coupon bond.

Second, thinking of duration in terms of years makes it difficult for managers and their clients to understand the duration of some complex securities. Here are a few examples. For a mortgage-backed security that is an interest-only security (i.e., receives coupons but not principal repayment) the duration is negative. What does a negative number, say, −4 mean?

In terms of our interpretation as a percentage price change, it means that when rates change by 100 basis points, the price of the bond changes by about 4% but the change is in the same direction as the change in rates.

As a second example, consider an inverse floater created in the collateralized mortgage obligation (CMO) market. The underlying collateral for such a security might be loans with 25 years to final maturity. However, an inverse floater can have a duration that easily exceeds 25.

This does not make sense to a manager or client who uses a measure of time as a definition for duration.

As a final example, consider derivative instruments, such as an option that expires in one year. Suppose that it is reported that its duration is 60. What does that mean? To someone who interprets duration in terms of time, does that mean 60 years, 60 days, 60 seconds? It doesn't mean any of these. It simply means that the option tends to have the price sensitivity to rate changes of a 60-year zero-coupon bond.

Forget First Derivatives and Temporal Definitions

The bottom line is that one should not care if it is technically correct to think of duration in terms of years (volatility of a zero-coupon bond) or in terms of first derivatives. There are even some who interpret duration in terms of the ''half life'' of a security. Subject to the limitations that we will describe as we proceed in this book, duration is the measure of a security's price sensitivity to changes in yield. We will fine tune this definition as we move along.

Users of this interest rate risk measure are interested in what it tells them about the price sensitivity of a bond (or a portfolio) to changes in interest rates. Duration provides the investor with a feel for the dollar price exposure or the percentage price exposure to potential interest rate changes. Try the following definitions on a client who has a portfolio with a duration of 4 and see which one the client finds most useful for understanding the interest rate risk of the portfolio when rates change:

Definition 1: The duration of 4 for your portfolio indicates that the portfolio's value will change by approximately 4% if rates change by 100 basis points.

Definition 2: The duration of 4 for your portfolio is the first derivative of the price function for the bonds in the portfolio.

Definition 3: The duration of 4 for your portfolio is the weighted average number of years to receive the present value of the portfolio's cash flows.

Definition 1 is clearly preferable. It would be ridiculous to expect clients to understand the last two definitions better than the first.

Moreover, interpreting duration in terms of a measure of price sensitivity to interest rate changes allows a manager to make comparisons between bonds regarding their interest rate risk under certain assumptions.

Portfolio Duration

A portfolio's duration can be obtained by calculating the weighted average of the duration of the bonds in the portfolio. The weight is the proportion of the portfolio that a security comprises. Mathematically, a portfolio's duration can be calculated as follows:

$$w_1 D_1 + w_2 D_2 + w_3 D_3 + \ldots + w_K D_K$$

where

w_i = market value of bond i/market value of the portfolio
D_i = duration of bond i
K = number of bonds in the portfolio

To illustrate this calculation, consider the following 3-bond portfolio in which all three bonds are option free:

Bond	Price ($)	Yield (%)	Par amount owned	Market value	Duration
10% 5-year	100.0000	10	$4 million	$4,000,000	3.861
8% 15-year	84.6275	10	5 million	4,231,375	8.047
14% 30-year	137.8586	10	1 million	1,378,586	9.168

In this illustration, it is assumed that the next coupon payment for each bond is exactly six months from now (i.e., there is no accrued interest). The market value for the portfolio is $9,609,961. Since each bond is option free, modified duration can be used. The market price per $100 par value of each bond, its yield, and its duration are given below:

In this illustration, K is equal to 3 and:

w1 = $4, 000, 000/$9, 609, 961 = 0.416 D1 = 3.861

w2 = $4, 231, 375/$9, 609, 961 = 0.440 D2 = 8.047

w3 = $1, 378, 586/$9, 609, 961 = 0.144 D3 = 9.168

The portfolio's duration is:

0.416(3.861) + 0.440(8.047) + 0.144(9.168) = 6.47

A portfolio duration of 6.47 means that for a 100 basis point change in the yield for each of the three bonds, the market value of the portfolio will change by approximately 6.47%.

But keep in mind, the yield for each of the three bonds must change by 100 basis points for the duration measure to be useful. (In other words, there must be a parallel shift in the yield curve.) This is a critical assumption and its importance cannot be overemphasized.

CONVEXITY ADJUSTMENT

The duration measure indicates that regardless of whether interest rates increase or decrease, the approximate percentage price change is the same. However, as we noted earlier, this is not consistent with Property 3 of a bond's price volatility. Specifically, while for small changes in yield the percentage price change will be the same for an increase or decrease in yield, for large changes in yield this is not true. This suggests that duration is only a good approximation of the percentage price change for small changes in yield.

We demonstrated this property earlier using a 9% 20-year bond selling to yield 6% with a duration of 10.66. For a 10 basis point change in yield, the estimate was accurate for both an increase or decrease in yield. However, for a 200 basis point change in yield, the approximate percentage price change was off considerably.

The reason for this result is that duration is in fact a first (linear) approximation for a small change in yield.9 The approximation can be improved by using a second approximation.

This approximation is referred to as the ''convexity adjustment.'' It is used to approximate the change in price that is not explained by duration.

The formula for the convexity adjustment to the percentage price change is:

$$\text{Convexity adjustment to the percentage price change} = C \times (\Delta y_*)^2 \times 100 \qquad (4)$$

where $\Delta y_* =$ the change in yield for which the percentage price change is sought and

$$C = \frac{V_+ + V_- - 2V_0}{2V_0(\Delta y)^2} \qquad (5)$$

The notation is the same as used in equation (1) for duration.[10]

For example, for our hypothetical 9% 20-year bond selling to yield 6%, we know from Section IV A that for a 20 basis point change in yield ($\Delta y = 0.002$):

$$V_0 = 134.6722,\ V- = 137.5888, \text{and } V_+ = 131.8439$$

Substituting these values into the formula for C:

$$C = \frac{131.8439 + 137.5888 - 2(134.6722)}{2(134.6722)(0.002)^2} = 81.95$$

Suppose that a convexity adjustment is sought for the approximate percentage price change for our hypothetical 9% 20-year bond for a change in yield of 200 basis points. That is, in equation (4), Δy_* is 0.02. Then the convexity adjustment is

$$81.95 \times (0.02)^2 \times 100 = 3.28\%$$

If the yield decreases from 6% to 4%, the convexity adjustment to the percentage price change based on duration would also be 3.28%.

The approximate percentage price change based on duration and the convexity adjustment is found by adding the two estimates. So, for example, if yields change from 6% to 8%, the estimated percentage price change would be:

Estimated change using duration =−21.32%

Convexity adjustment = +3.28%

Total estimated percentage price change =−18.04%

The actual percentage price change is −18.40%.

For a decrease of 200 basis points, from 6% to 4%, the approximate percentage price change would be as follows:

Estimated change using duration =+21.32%

Convexity adjustment = +3.28%

Total estimated percentage price change =+24.60%

The actual percentage price change is +25.04%. Thus, duration combined with the convexity adjustment does a better job of estimating the sensitivity of a bond's price change to large changes in yield (i.e., better than using duration alone).

Positive and Negative Convexity Adjustment

Notice that when the convexity adjustment is positive, we have the situation described earlier that the gain is greater than the loss for a given large change in rates. That is, the bond exhibits positive convexity. We can see this in the example above. However, if the convexity adjustment is negative, we have the situation where the loss will be greater than the gain. For example, suppose that a callable bond has an effective duration of 4 and a convexity adjustment for a 200 basis point change of −1.2%.

The bond then exhibits the negative convexity property illustrated in the previous Exhibit 11. The approximate percentage price change after adjusting for convexity is:

Estimated change using duration =−8.0%

Convexity adjustment = −1.2%

Total estimated percentage price change =−9.2%

For a decrease of 200 basis points, the approximate percentage price change would be as follows:

Estimated change using duration =+8.0%

Convexity adjustment = −1.2%

Total estimated percentage price change =+6.8%

Modified and Effective Convexity Adjustment

The prices used in computing C in equation (4) to calculate the convexity adjustment can be obtained by assuming that, when the yield changes, the expected cash flows either do not change or they do change. In the former case, the resulting convexity is referred to as modified convexity adjustment. (Actually, in the industry, convexity adjustment is not qualified by the adjective ‘‘modified.’’) In contrast, effective convexity adjustment assumes that the cash flows change when yields change. This is the same distinction made for duration.

As with duration, there is little difference between a modified convexity adjustment and an effective convexity adjustment for option-free bonds. However, for bonds with embedded options, there can be quite a difference between the calculated modified convexity adjustment and an effective convexity adjustment. In fact, for all option-free bonds, either convexity adjustment will have a positive value. For bonds with embedded options, the calculated effective convexity adjustment can be negative when the calculated modified convexity adjustment is positive.